Large N_c QCD, Harmonic Sums and Holography

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Introduction

- **What is the Effective Field Theory of QCD at Long–Distances ?**

- **How does QCD fix the couplings of the Chiral Lagrangian of the Nambu-Goldstone modes of the spontaneously broken chiral–$\mathrm{SU}(3)$ symmetry ?**
  Weinberg '79, Gasser-Leutwyler '85

Quick Personal Historical Overview

- The Constituent Chiral Quark Model *Manohar and Georgi '84, ..., Weinberg 10*
- The Extended Nambu Jona-Lasinio Model *Nambu Jona-Lasinio '61, ... Bruno-Bijnens, de Rafael '93*

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Introduction: Large-N_c QCD *t'Hooft 74*

In Large-N_c QCD the Hadronic Spectrum consists of an <u>infinite</u> number of Narrow States *Witten '79*

Any Relation to the MG and/or ENJL–Models?

- When unconfining $Q\bar{Q}$ pairs in ENJL spectral functions are removed (by appropriate series of local counterterms) what results is an effective Resonance Chiral Lagrangian of three narrow states V, A and S *Peris-Perrottet-de Rafael '98*
- Resonance Chiral Lagrangians *Ecker et al '89* can then be viewed as simplified versions of the Large–N_c QCD Hadronic Lagrangian, limited to its dominant low states

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Introduction: Observations

Two Observations

- Couplings of the Effective Chiral Lagrangian of the Strong Interactions are coefficients of the Taylor expansion of appropriate QCD Green's Functions.
- Couplings of the Effective Chiral Lagrangian of the Electroweak Hadronic Interactions are integrals over euclidean momenta of appropriate QCD Green's Functions.
  Matching of the short and long–distances required.

The MG and ENJL Models fail to incorporate short–distance behaviour.
Replaced by more direct approach where Green's functions are approximated by finite number of narrow states:

- Short–Distance behaviour (OPE) is satisfied.
- Long–Distance behaviour (Chiral Lagrangian) are satisfied.

Many Successful Phenomenological Applications

M. Knecht, S. Peris , M. Perrottet, Th. Hambye, A. Pich, G. Ecker, J. Prades, B. Moussalam, A. Nyffeler, O. Catá, M. Golterman, E. de Rafael...

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Generic Properties of Large–N_c QCD

In large–N_c QCD

Two-Point Functions of local color singlet currents are **Harmonic Sums**.

Harmonic Sums are characterized by
Base Function and **Dirichlet Series**

Dirichlet Series

$$\Sigma(s) = \sum_{n=1}^{\infty} \lambda_n \, \rho_n^{-s}$$

The λ_n are called *Amplitudes* The ρ_n are called *Frequencies*

The simplest case of a *Dirichlet Series* is the *Riemann zeta function*:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes}(p)} \frac{1}{1-p^{-s}}\,, \quad \mathrm{Re}(s) > 1\,,$$

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Example: The Adler Function in Large N_c QCD

With $\Pi(q^2)$ the Vector-Vector Correlation Function in QCD
The Adler Function ($Q^2 = -q^2 \geq 0$):

$$\mathcal{A}(Q^2) = \int_0^\infty dt \frac{Q^2}{(t+Q^2)^2} \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$

The Spectral Function in Large–N_c QCD:

$$\frac{1}{\pi} \mathrm{Im}\,\Pi(t) = \sum_{n=1}^{\infty} \gamma_n\, M_n^2\, \delta(t - M_n^2)$$

Setting

$$z = \frac{M^2}{Q^2} \quad \text{and} \quad \mu_n = \frac{M_n^2}{M^2} \quad \text{with} \quad M^2 \equiv M_1^2$$

the Adler Function becomes a sum of the type:

$$\mathcal{A}(Q^2) = \sum_{n=1}^{\infty} \gamma_n \frac{\mu_n z}{(1+\mu_n z)^2}$$

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This is an Harmonic Sum

Recall: $z = \frac{M^2}{Q^2}$ and $\mu_n = \frac{M_n^2}{M^2}$ with $M^2 \equiv M_1^2$

$\therefore$ In Large-N_c QCD the Adler Function:

$$\mathcal{A}(Q^2) = \sum_{n=1}^{\infty} \gamma_n \frac{\mu_n z}{(1+\mu_n z)^2}$$

The Adler Function in Large-N_c QCD is a typical Harmonic Sum

$$\mathcal{A}(z) = \gamma \sum_{n=1}^{\infty} \lambda_n \, g_{\text{Adler}}(\mu_n z)$$

$$\gamma \equiv \gamma_1 \,, \qquad \lambda_n = \frac{\gamma_n}{\gamma} \,; \qquad \gamma = \frac{3}{2\pi\alpha^2}\frac{1}{M_\rho}\Gamma_{\rho\to e^+e^-}$$

The *poles* are the *frequencies* μ_n

The *residues* are the *amplitudes* λ_n

The *Base Function* is:

$$g_{\text{Adler}}(x) = \frac{x}{(1+x)^2}$$

The *Dirichlet Series* is:

$$\Sigma(s) = \sum_{n=1}^{\infty} \lambda_n \mu_n^{-s}$$

Harmonic Sums have *factorizable* Mellin-Transforms:

$$\mathcal{M}[\mathcal{A}(z)](s) = \gamma\, \mathcal{M}[g_{\mathrm{Adler}}(z)](s)\, \Sigma(s)\,, \qquad \underbrace{\Sigma(s) = \sum_{n=1}^{\infty} \lambda_n \mu_n^{-s}}_{\text{Mellin of Spectral Function}}$$

$$\mathcal{M}[g_{\mathrm{Adler}}(z)](s) = \int_0^{\infty} dz\, z^{s-1}\, g_{\mathrm{Adler}}(z) = \Gamma(1+s)\Gamma(1-s)$$

Conclusion

In Large–N_c QCD, the Adler Function has a Mellin-Barnes representation:

$$\mathcal{A}(z) = \frac{\gamma}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, z^{-s}\, \Sigma(s)\, \Gamma(1+s)\Gamma(1-s) \qquad \text{with} \qquad z = \frac{M^2}{Q^2}\,.$$

The Dirichlet Series $\Sigma(s)$ governs the Dynamics of the Large-N_c

The *Inverse Mapping Theorem* relates *coefficients* of physical asymptotic series in z to *residues* of *singular expansion* of integrand:

z-small (*OPE*) $\Leftrightarrow$ *left of the fundamental strip*
z-large (*Chiral Expansion*) $\Leftrightarrow$ *right of fundamental strip*

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Choice of a Dirichlet Series corresponds to Model Large–N_c

- Example: Equally Spaced Spectrum
  $\frac{1}{\pi}\mathrm{Im}\Pi(t) = A\,\sigma^2 \sum_{n=0}^{\infty} \delta(t - M_0^2 - n\sigma^2)$, results in the Dirichlet Series:

$$\Sigma^{(\mathrm{es})}(s) = \left(1 + n\frac{\sigma^2}{M_0^2}\right)^{-s} , \quad \text{Hurwitz Function}$$

- *In the absence of knowledge from first principles of the underlying Large-N_c QCD Dirichlet Series , one can proceed by making an ansatz based on known properties of the two–point function one is considering: OPE (short–distances) and χPT (long–distances).*

COMMENT

This framework is to be contrasted with the fashionable "AdS/CFT Large-N_c QCD inspired models".

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The $\Pi_{\rm LR}(Q^2)$ Correlation Function in QCD

In QCD in the chiral limit:

$$(q^\mu q^\nu - g^{\mu\nu} q^2)\Pi_{\rm LR}(Q^2) = 2i \int_0^\infty d^4x\, e^{iq\cdot x} \langle 0 \mid T\left(L^\mu(x) R^\nu(0)^\dagger\right) \mid 0\rangle$$

$$L^\mu(x) = \bar{d}(x)\gamma^\mu \frac{1}{2}(1-\gamma_5)u(x) \quad \text{and} \quad \mathrm{R}^\mu(\mathrm{x}) = \bar{\mathrm{d}}(\mathrm{x})\gamma^\mu \frac{1}{2}(1+\gamma_5)\mathrm{u}(\mathrm{x})\,.$$

Mellin-Barnes Representation

$$-Q^2\Pi_{\rm LR}(Q^2) = -\int_0^\infty dt\, \frac{Q^2}{Q^2+t}\,\frac{1}{\pi}\mathrm{Im}\Pi_{\rm LR}(t)$$

$$= -\int_0^\infty dt\, \frac{1}{2\pi i}\int_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{t}{Q^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\frac{1}{\pi}\mathrm{Im}\Pi_{\rm LR}(t)$$

$$= \frac{F_0^2}{M_\rho^2} - \frac{1}{2\pi i}\int_{c_s - i\infty}^{c_s + i\infty} ds\, \mathcal{M}(s) \left(\frac{M_\rho^2}{Q^2}\right)^{-s} \Gamma(s)\Gamma(1-s)$$

with $\mathcal{M}(s)$ the Mellin Transform of the Spectral Function (without pion pole)

$$\mathcal{M}(s) = \int_0^\infty \frac{dt}{t}\left(\frac{M_\rho^2}{t}\right)^{s-1} \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}_{\rm LR}(t)$$

The $\Pi_{LR}(Q^2)$ Correlation Function in QCD

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$$\begin{aligned} -Q^2\Pi_{LR}(Q^2) &= -\int_0^\infty dt\, \frac{Q^2}{Q^2+t}\, \frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) \\ &= -\int_0^\infty dt\, \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds\, \left(\frac{t}{Q^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) \\ &= \frac{F_0^2}{M_\rho^2} - \frac{1}{2\pi i} \int_{c_s-i\infty}^{c_s+i\infty} ds\, \mathcal{M}(s) \left(\frac{M_\rho^2}{Q^2}\right)^{-s} \Gamma(s)\Gamma(1-s) \end{aligned}$$

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Large–N_c Model of the LR–Correlation Function

Large–N_c Model of the LR–Spectral Function

$$\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -F_0^2\delta(t) + 2F_V^2\sum_{n=0}^{\infty}\delta(t-M_V^2-n\sigma^2) - 2F_A^2\sum_{n=0}^{\infty}\delta(t-M_A^2-n\sigma^2)$$

Implementing 1st and 2nd Weinberg Sum Rules results in

$$-\frac{Q^2}{F_0^2}\Pi_{\rm LR}(Q^2) = 1 - \frac{g_A}{1-g_A}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(1+\frac{1}{g_A}\right)^{1-s}\times$$

$$\left\{\zeta\left(s,\frac{g_A}{1+g_A}\right)-\zeta\left(s,\frac{1}{1+g_A}\right)\right\}\left(\frac{M_V^2}{Q^2}\right)^{-s}\Gamma(s)\Gamma(1-s)\,,$$

Here $g_A = \frac{M_V^2}{M_A^2}$, $F_V^2 = F_A^2 = \frac{F_0^2}{2}\,\frac{1+g_A}{1-g_A}$, and $\sigma^2 = \mathrm{M}_\mathrm{V}^2\left(1+\frac{1}{\mathrm{g_A}}\right)$.

$$\zeta(s,v) = \sum_{n=0}^{\infty}\frac{1}{(n+v)^s}\,,\quad \mathrm{Re}s>0 \quad \text{and} \quad \mathrm{v}\neq 0,-1,-2,\cdots,$$

with known analytic continuation, $\zeta(-m,v) = -\frac{B_{m+1}(v)}{m+1}$, $m = 0,1,2,3,\ldots$

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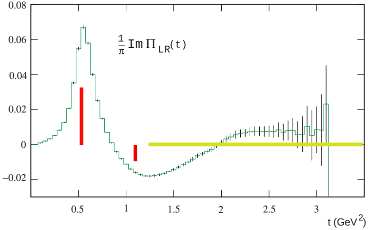
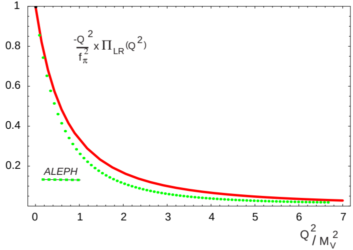
Plots of the $\Pi_{LR}(Q^2)$ Two–Point Function (Data and MHA)

Data from LEP-ALEPH

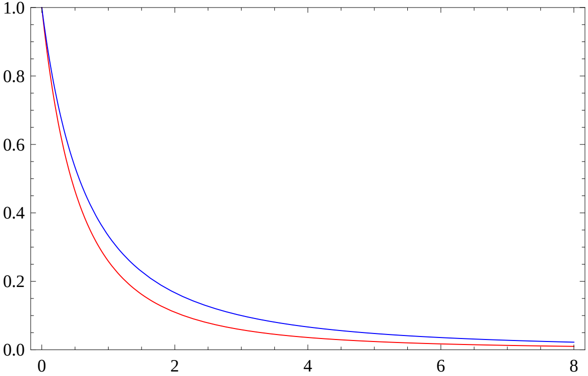




Plot of $\Pi_{LR}(Q^2)$ in the Euclidean

Plots of the $\Pi_{LR}(Q^2)$ Two–Point Function (Model and MHA)

Plot of $-\frac{Q^2}{F_0^2}\Pi_{LR}(Q^2)$ (Hurwitz Large–N_c Model)
versus $\frac{Q^2}{M_V^2}$ (for $g_A = 1/2$)



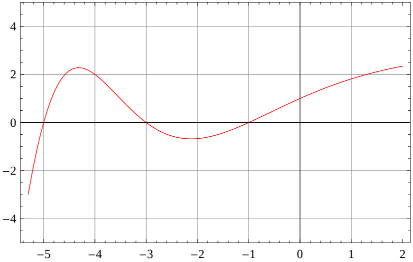
The blue curve corresponds to the *Minimal Hadronic Ansatz* Large–N_c Model

Plots of the Mellin Transform of the Spectral Function (Model and MHA)

$$\mathcal{M}(s) = \int_0^\infty \frac{dt}{t}\left(\frac{M_V^2}{t}\right)^{s-1}\ \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}_{\mathrm{LR}}(t)$$

In the Hurwitz-like model of Large-N_c QCD

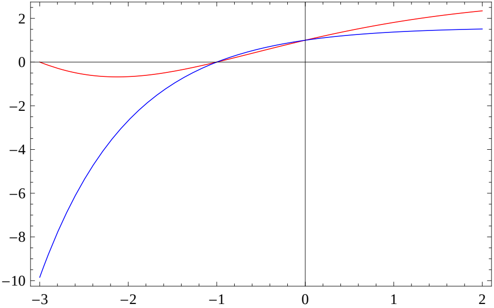
$$\mathcal{M}_{\mathrm{HM}}(s) = \frac{g_A}{1-g_A}\left(1+\frac{1}{g_A}\right)^{1-s}\left[\zeta\left(s,\frac{g_A}{1+g_A}\right)-\zeta\left(s,\frac{1}{1+g_A}\right)\right].$$



Mellin Transform versus s normalized to $\frac{F_0^2}{M_V^2}$ *for* $g_A = \frac{1}{2}$

Comments on Higher Dimension Condensates

Mellin Transforms of Spectral Functions
MHA model and Hurwitz-like model



The Son–Yamamoyo Relation

Son, Yamamoto '10

In theories whose gravity dual is described by the Yang-Mills-Chern-Simons theory with chiral symmetry broken by boundary conditions in the infrared:

$$w_L(Q^2) - 2w_T(Q^2) = -\frac{2N_c}{f_\pi^2}\Pi_{\mathrm{LR}}(Q^2)$$

$w_L(Q^2)$ and $w_T(Q^2)$ are the longitudinal and transverse functions of the VVA triangle of electroweak hadronic currents in a specific kinematic configuration

$$Q^2\left[w_L(Q^2) - 2w_T(Q^2)\right] = \frac{16\pi^2}{\sqrt{3}}\int d^4x \int d^4y e^{iq\cdot x}(x-y)_\lambda \epsilon^{\mu\nu\rho\lambda}\langle 0|\hat{T}\left\{L^3_\mu(x)V^3_\nu(y)R^8_\rho(0)\right\}|0\rangle$$

$$L^3_\mu(x) = \bar{\psi}(x)\frac{\lambda_3}{2}\gamma_\mu\frac{1-\gamma_5}{2}\psi(x)\,, \quad R^8_\rho(x) = \bar{\psi}(0)\frac{\lambda_8}{2}\gamma_\rho\frac{1+\gamma_5}{2}\psi(0)\,, \quad V^3_\nu(y) = \bar{\psi}(y)\frac{\lambda_3}{2}\gamma_\nu\psi(y)$$

Can this be true in Large–N_c QCD ?

Knecht, Peris, de Rafael '11

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Knecht, Peris, de Rafael '11

Short and Long Distances Behaviours

- Short–Distance Behaviour of $\Pi_{LR}(Q^2)$

Shifmann, Vainshtein, Zakharov '79

$$\Pi_{LR}(Q^2) \underset{Q^2\to\infty}{\sim} -4\pi^2\left(\frac{\alpha_s}{\pi}+\mathcal{O}(\alpha_s^2)\right)\langle\bar\psi\psi\rangle^2\frac{1}{Q^6}+\mathcal{O}\left(\frac{1}{Q^8}\right)$$

- Long–Distance Behaviour of $\Pi_{LR}(Q^2)$

Gasser, Leutwyler '85; Amoros, Bijnens, Talavera '00

$$-Q^2\Pi_{LR}(Q^2) \underset{Q^2\to 0}{\sim} f_\pi^2+4L_{10}Q^2+8C_{87}Q^4+\mathcal{O}(Q^6)$$

- The *Adler-Bell-Jackiw* anomaly fixes $w_L(Q^2)=2\frac{N_c}{Q^2}$ at all Q^2
- Short–Distance Behaviour $w_T(Q^2)$

Knecht, Peris, Perriottet, de Rafael '02; Vainshtein '03

$$w_T(Q^2) \underset{Q^2\to\infty}{\sim} \frac{N_c}{Q^2}-32\pi^4\left(\frac{\alpha_s}{\pi}+\mathcal{O}(\alpha_s^2)\right)\langle\bar\psi\psi\rangle\,\Pi_{VT}(0)\frac{1}{Q^6}+\mathcal{O}\left(\frac{1}{Q^8}\right)$$

- Long–Distance Behaviour $w_T(Q^2)$

Knecht, Peris, Perriottet, de Rafael '02

$$w_T(Q^2) \underset{Q^2\to 0}{\sim} 128\pi^2 C_{22}^W+\mathcal{O}(Q^2)$$

Short and Long Distances Behaviours of the Son-Yamamoto Relation

$$w_L(Q^2) - 2w_T(Q^2) = -\frac{2N_c}{f_\pi^2}\Pi_{\mathrm{LR}}(Q^2)$$

$$\Pi_{\mathrm{LR}}(Q^2) \underset{Q^2 \to \infty}{\sim} -4\pi^2 \left(\frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right) \langle\bar{\psi}\psi\rangle^2 \frac{1}{Q^6} + \mathcal{O}\left(\frac{1}{Q^8}\right)$$

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$$w_T(Q^2) \underset{Q^2 \to 0}{\sim} 128\pi^2 C_{22}^W + \mathcal{O}(Q^2) \quad \text{and} \quad \mathrm{w_L}(\mathrm{Q}^2) = 2\frac{\mathrm{N_c}}{\mathrm{Q}^2}$$

- At short–distances: $\Pi_{\mathrm{VT}}(0) = \frac{1}{8\pi^2}\frac{N_c}{f_\pi^2}\langle\bar{\psi}\psi\rangle$
- At long–distances: $C_{22}^W = \frac{N_c}{32 f_\pi^2} L_{10}$

The Son-Yamamoto Relation in perturbative QCD

$$w_L(Q^2) - 2w_T(Q^2) = -\frac{2N_c}{f_\pi^2}\Pi_{\rm LR}(Q^2)$$

This requires a non-zero quark mass m and dimensional regularization $d = 4$

Knecht-Peris-de Rafael '11

- The left–hand side *Melnikov '06*

$$\lim_{\epsilon\to 0}\left[w_L(Q^2;m,\epsilon) - 2\,w_T(Q^2;m,\epsilon)\right] = \frac{N_c}{Q^2}\left(\frac{N_c^2-1}{2N_c}\right)\frac{\alpha_s}{\pi}\frac{m^2}{Q^2}\left(2\log\frac{Q^2}{m^2}+1\right) + \mathcal{O}(m^4/Q^4,\alpha_s^2)$$

$$\therefore \quad \lim_{m\to 0}\lim_{\epsilon\to 0}\left[w_L(Q^2;m,\epsilon) - 2w_T(Q^2;m,\epsilon)\right] = 0$$

- The right–hand side $\Pi^{\mu\nu}_{\rm LR}(q) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi^{(1)}_{\rm LR}(Q^2) + q^\mu q^\nu \Pi^{(0)}_{\rm LR}(Q^2)$

$$\Pi^{\mu\nu}_{\rm LR}(q) = -\frac{N_c}{4\pi^2}m^2 g_{\mu\nu}\left\{\frac{2}{\epsilon} - \gamma_{\rm E} + \log 4\pi - \log\frac{m^2}{\nu^2} + 2 + \sqrt{1+\frac{4m^2}{Q^2}}\log\left[\frac{\sqrt{1+\frac{4m^2}{Q^2}}-1}{\sqrt{1+\frac{4m^2}{Q^2}}+1}\right]\right\}$$

$$\therefore \quad \lim_{m\to 0}\lim_{\epsilon\to 0}\left[-\frac{2N_c}{f_\pi^2}\Pi_{\rm LR}(Q^2)\right]\Big|_{\rm pQCD} = 2\frac{N_c}{Q^2}$$

The Son–Yamamoto Holographic Relation is not a property of QCD

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The Son–Yamamoto Holographic Relation is not a property of QCD

The Riemann Zeros and Sum Rules

THE RIEMANN ZETA FUNCTION

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{primes}} \frac{1}{1-p^{-s}}\,, \quad \mathrm{Re(s)} > 1\,.$$

Work with the logarithmic derivative:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{\text{primes } p} \log(p) \sum_{k=1}^{\infty} p^{-ks} = \underbrace{\sum_{n=1}^{\infty} \Lambda(n)\, n^{-s}}_{\text{Dirichlet Series}}\,, \qquad \mathrm{Re(s)} > 1\,,$$

The $\Lambda(n)$ are the Von Mangoldt Amplitudes:

$$\Lambda(n) = \begin{cases} \log(p)\,, & \text{if } n = p^k \text{ is a power of a single prime} \\ 0\,, & \text{otherwise} \end{cases}\,.$$

Analytic continuation (from Hadamard's product formula):

$$\Sigma_{\mathrm{VonM}}(s) = -\frac{\zeta'(s)}{\zeta(s)} = \log\frac{1}{2\pi} + \frac{s}{s-1} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \sum_{\rho} \frac{s}{\rho(s-\rho)}\,,$$

where ρ are the *non-trivial zeros* of the Riemann Zeta Function $\mathrm{Re}(\rho) \in]0,1[$
The Riemann Hypothesis $\Rightarrow \rho = 1/2 \pm i\eta$

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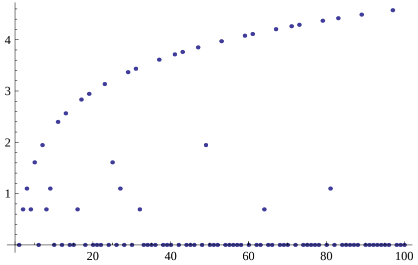
The Von Mangoldt Spectral Function

QUESTION

What are the Properties of a QFT-Like Green's Function which has as a Spectral Function:

$$\frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{VonM}}(t) = \sum_{n=1}^{\infty} \Lambda(n)\, nM^2\ \delta(t - nM^2)\,,$$

i.e., the one which generates the Von Mangoldt Dirichlet Series?



The Von Mangoldt values $\Lambda(n)$ for the first 100 integers

The Von Mangoldt's Explicit Formula

$$\Psi_{\rm VonM}(x)=\frac{1}{2}\left(\sum_{n<x}\Lambda(n)+\sum_{n\leq x}\Lambda(n)\right)\,.$$

Quite remarkably, there is a simple explicit formula for this function in terms of the non–trivial zeros of $\zeta(s)$:

$$\Psi_{\rm VonM}(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{1}{2}\log\left(1-x^{-2}\right)-\log(2\pi)\,.$$

Notice that the Von Mangoldt explicit formula, within the framework of Quantum Field Theory, is nothing but what we call a Finite Energy Sum Rule:

$$\int_{0}^{s_0=xM^2}\frac{dt}{t}\frac{1}{\pi}{\rm Im}\Pi_{\rm VonM}(t)=\Psi_{\rm VonM}(x)\,.$$

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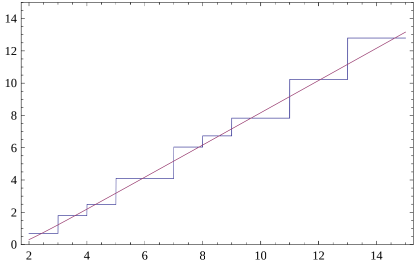
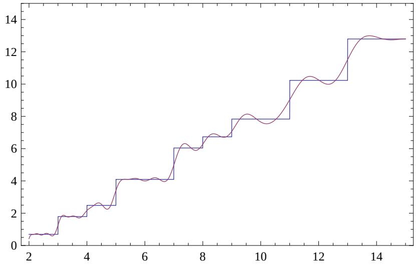
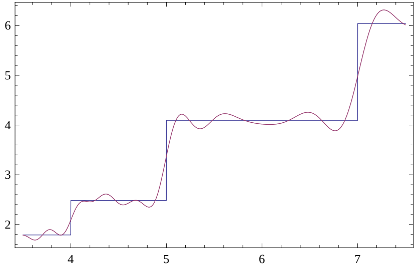


The Von Mangoldt Function $\Psi_{\mathrm{VonM}}(x)$ *compared to the explicit formula without the contribution of the non–trivial zeros (the continuous line) for* $2 \leq x \leq 15$.



The Von Mangoldt Function $\Psi_{\mathrm{VonM}}(x)$ *compared to the explicit formula including the contribution of the first ten non–trivial zeros (the continuous line) for* $2 \leq x \leq 15$.



The Von Mangoldt Function $\Psi_{\mathrm{VonM}}(x)$ *compared to the explicit formula including the contribution of the first twenty five non–trivial zeros (the continuous line) for* $3.5 \leq x \leq 7.5$.

The Von Mangoldt Harmonic Sum

Mellin-Barnes Representation of the Von Mangoldt Harmonic Sum

$$\mathcal{P}_{\mathrm{VonM}}(z) = \int_0^\infty dt\, \frac{Q^4}{(t+Q^2)^3}\, \frac{1}{\pi}\mathrm{Im}\Pi_{\mathrm{VonM}}(t) = \sum_{n=1}^{\infty} \Lambda(n) \frac{nz}{(1+nz)^3}\,, \quad z = \frac{M^2}{Q^2}$$

$$= \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} ds\, z^{-s}\, \Sigma_{\mathrm{VonM}}(s)\, \Gamma(s+1)\Gamma(2-s)\,, \quad c = \mathrm{Re}(s) \in\,]+1,+2[$$

$$\Sigma_{\mathrm{VonM}}(s) = \log\frac{1}{2\pi} + \frac{s}{s-1} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \sum_{\rho} \frac{s}{\rho(s-\rho)}$$

- Leading $s=1$ singularity $\Rightarrow \frac{Q^2}{M^2}$ term
- Singularities at $s=-1,-2,-3,\cdots$ from $\Gamma(s+1)$ and $s=-2,-4,-6\cdots$ from $\Gamma(s+1)$ and trivial zeros

Terms $\mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n+1}$ and $\mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n} \log\frac{M^2}{Q^2}\,,\quad \mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n}$

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- Leading $s = 1$ singularity $\Rightarrow \frac{Q^2}{M^2}$ term
- Singularities at $s = -1, -2, -3, \cdots$ from $\Gamma(s+1)$ and $s = -2, -4, -6\cdots$ from $\Gamma(s+1)$ and trivial zeros

$$\text{Terms} \quad \mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n+1} \quad \text{and} \quad \mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n} \log\frac{M^2}{Q^2}\,, \quad \mathcal{O}\left(\frac{M^2}{Q^2}\right)^{2n}$$

Singularities from the Non-Trivial Zeros

Mellin-Barnes Representation of the Von Mangoldt Harmonic Sum

$$\mathcal{P}_{\mathrm{VonM}}\left(\frac{M^2}{Q^2}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \left(\frac{M^2}{Q^2}\right)^{-s} \Sigma_{\mathrm{VonM}}(s)\,\Gamma(s+1)\Gamma(2-s)\,,$$

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The *non–trivial zeros of the Riemann zeta function* generate *non–power terms* in the asymptotic expansion at large Q^2 modulated by an *oscillating behaviour* in Q^2. These terms appear in pairs: $\mathcal{O}\left(\frac{Q^2}{M^2}\right)^{|\rho|}$ and $\mathcal{O}\left(\frac{Q^2}{M^2}\right)^{1-|\rho|}$, unless the *Riemann Hypothes* is true, in which case *the only allowed non–power term* is:

$$-\sum_{\eta}\frac{1+4\eta^2}{2}\,\frac{\pi}{\cosh\pi\eta}\sqrt{\frac{Q^2}{M^2}}\cos\left(\eta\log\frac{M^2}{Q^2}\right)$$

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Conclusion

The Riemann Hypothesis is equivalent to the existence of a unique non–power term:

$$-\sum_{\eta} \frac{1+4\eta^2}{2} \frac{\pi}{\cosh \pi\eta} \sqrt{\frac{Q^2}{M^2}} \cos\left(\eta \log \frac{M^2}{Q^2}\right) ,$$

in the short–distance expansion of the two–point function associated to the Von Mangoldt Spectral Function.