Beyond SM - Exercices

1 Groups, spinors and gauge transformations

1. For Lie Groups, a group element arbitrarily close to the identity can be expanded as:

$$U = I + i\alpha^a T^a \tag{1}$$

where α^a are real parameters and T^a are the generators. The latter form a Lie algebra that is defined though the commutation relations:

$$[T^a, T^b] = i f^{abc} T^c \tag{2}$$

where f^{abc} are the structure constants and are real.

   (a) For U a group element of SU(N) determine the properties of the generators (hermitian, etc) and, for $N \times N$ transformations (corresponding to the fundamental representation) determine the number of the group generators (corresponding to the dimension of the algebra).

   (b) Check that for SU(2), the generators $T^a = \sigma^a/2$, where σ^a are the Pauli Matrices, satisfy the commutation relations above with $f^{abc} = \epsilon^{abc}$.

   (c) Check that for SU(2), the representation of dimension 2 satisfy $-T^{a*} = UT^aU^\dagger$ with $U = i\sigma_2$. This representation is said to be pseudo-real.[1]

   (d) Consider an SU(2) lepton doublet,

$$L_e = (\nu_{eL}\, e_L)^T\,. \tag{3}$$

   How does this doublet transforms under an SU(2) transformation described by (1) with $\alpha^3 = \alpha$, and $\alpha^1 = \alpha^2 = 0$.

2. We are now going to look at the Lorentz transformations acting on spinors.

   (a) The 4 dimensional matrices $S^{\mu\nu} = i/4[\gamma^\mu, \gamma^\nu]$ provide a representation of the Lorentz algebra. Give the form of S^{0i} and S^{ij} in terms of the Pauli matrices using the Weyl representation of the Dirac matrices γ^μ.

   (b) Considering that a Dirac Spinor transform as $\Psi \to \exp(-i\omega_{\mu\nu}S^{\mu\nu})\Psi$, where $\omega_{\mu\nu}$ is ar antisymmetric tensor, check that the left and right components of the spinor $\Psi = (\psi_L\, \psi_R)^T$

      - transform in the same way when considering an infinitesimal rotation of angle $\omega_{12} = -\omega_{12} = \theta$ in the xy plane

[1] For a given representation R described with the generators T^a_R, the complex conjugate representation $\bar{R}$ with $T^a_{\bar{R}} = -T^{a*}_R$ is also a representation as it satisfy (2). A real representation satisfy $T^a_{\bar{R}} = T^a_R$ (or there is a unitary transformation V such that $\tilde{T}^a_R = V^{-1}T^a_R V$ satisfy $\tilde{T}^a_{\bar{R}} = \tilde{T}^a_R$). You can check that the Adjoint representations of SU(N) or the fundamental representations of SO(N) are real representations.

- transform differently under an infinitesimal boost of rapidity $\omega_{01} = -\omega_{10} = \beta$ in the x-direction.

This is directly related to the fact that the Dirac representation of the Lorentz group is reducible.

(c) Check that $i\sigma^2\chi_L^*$ transforms as ψ_R using the last item of Exercise 1

(d) Check that $\chi_L^T i\sigma^2\psi_L$ is a scalar under Lorentz transformations, conclude that $\bar{\Psi}\Psi$ with $\Psi = (\psi_L\,\psi_R)^T$ is a Lorentz scalar.

3. Considering the covariant derivative

$$D_\mu = \partial_\mu - igA_\mu\,, \tag{4}$$

the field strength can be defined as $[D_\mu, D_\nu] = -igF_{\mu\nu}$.

(a) For an abelian gauge group check that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) For a non abelian gauge group check that $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ with $F_{\mu\nu} = F^a_{\mu\nu}T^a$ and $A_\mu = A^a_\mu T^a$.

4. The kinetic term for and $SU(2)_L\times U(1)_Y$ lepton doublet $L_e = (L_{e1}\,L_{e2})^T$ with $L_{e1} = \nu_{eL}$ and $L_{e2} = e_L$ reads $\bar{L}_e \not{D} L_e$ with $D_\mu = \partial_\mu - ig'Y_{eL}B_\mu - igW_\mu$ and $W_\mu = W^a_\mu\frac{\sigma^a}{2}$. In the compact expression of this kinetic term, write explicitly the sum over SU(2), Lorentz and spinor indices.

2 Processes

In order to fix the notations, here we follow the notations of [1] describing a Dirac spinor as

$$\psi\,(x,t) = \sum_s \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \left(a_{k,s}u\,(k,s)\,e^{-ikx} + b^\dagger_{k,s}v\,(k,s)\,e^{ikx}\right)\,. \tag{5}$$

The sum runs over spin values s and $a^\dagger_{k,s}$ $\left(b^\dagger_{k,s}\right)$ creates a particle (antiparticle) of momentum $\vec{k}$ and spin s. The spinors u and v obey the Dirac equations

$$\begin{aligned}(\not{k} - m)u\,(k,s) &= 0\\ (\not{k} + m)v\,(k,s) &= 0\end{aligned} \tag{6}$$

with the sums:

$$\sum_s u\,(k,s)\,\bar{u}\,(k,s) = \not{k} + m, \quad \sum_s v\,(k,s)\,\bar{v}\,(k,s) = \not{k} - m \tag{7}$$

For a vector, we use:

$$V^\mu\,(x,t) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}} \left(a_{k,\lambda}\epsilon^\mu_{k,\lambda}e^{-ikx} + b^\dagger_{k,s}\epsilon^{\mu*}_{k,\lambda}e^{ikx}\right) \tag{8}$$

where the sum runs over the polarizations λ and the $\epsilon^\mu_{k,\lambda}$ are the polarization vectors that satisfy:

$$\epsilon^\mu_{k,\lambda}\epsilon_{k,\lambda'\,\mu} = -\delta_{\lambda\lambda'} \quad \text{and} \quad k_\mu\epsilon^\mu_{k,\lambda} = 0\,. \tag{9}$$

Also, in order to calculate decay widths, you can make use of the following expression for the decay width [2]:

$$d\Gamma(X \to ab) = \frac{1}{32\pi^2}|\mathcal{M}|^2 \frac{|\vec{p}_a|}{m_X^2} d\Omega \tag{10}$$

which is valid in the restframe of the particle X. $|\mathcal{M}|^2$ refers to the transition matrix squared summed (averaged) over final (initial) state polarization and spins and $\vec{p}_a$ and $d\Omega = d\phi_a d\cos\theta_a$ are the momentum and the solid angle of particle a respectively.

2.1 Decay of a gauge boson and number of ν_L families

1. Compute the decay width for the process:

$$W^- \to e^- \bar{\nu}_e \tag{11}$$

   neglecting electron and the neutrino masses. For that purpose, use the lagrangian for gauge boson-lepton interactions that were obtained in the lectures. You can also use of the following tools:

   (a) the sum over the polarization vectors of the on-shell W boson of 4-momentum k is given by $\sum_\lambda \epsilon^\mu_{k,\lambda}\epsilon^{\nu\,*}_{k,\lambda} = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2}$

   (b) tr(any odd nb. of γ's)=0

   (c) $\mathrm{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = 4\,\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\alpha\nu}g^{\mu\beta}\right)$

   (d) $\mathrm{tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^5\right) = -4i\,\epsilon^{\alpha\beta\mu\nu}$

   (e) $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ in Dirac and Weyl representations.

2. Compute the decay width for the process:

$$Z \to \bar{\nu}\nu\,. \tag{12}$$

   You can use the similitudes between the Lagrangians driving (13) and (12) for a rapid evaluation.
   This is of interest because the total decay width of the Z boson can be obtained analyzing the total cross-section for e^+e^- annihilation at the Z pole obtained at LEPI and SLAC. Compare the obtained decay width with the invisible decay width of Z obtained experimentally (see e.g. PDG) and deduce the number of families of active neutrinos in Nature.

2.2 Decay of SM scalar boson

1. Compute the tree level decay width for the processes:

$$h \to f\bar{f}, VV \tag{13}$$

   where f are SM fermions and $V = Z, W$ gauge bosons.

2. Could the $V^{(*)}V^*$ final state (with one or two offshell V boson) be the most important contribution to the h decay width for $m_h < 2m_V$? Argue why.

3. Draw the lowest order contributions to $h \to gg, \gamma\gamma, Z\gamma$.

2.3 Decay of the top quark and Goldstone bosons

1. Compute the decay width for the process:

$$t \to W^+ b \tag{14}$$

using the lagrangian for gauge boson-quarks interactions that were obtained in the lectures and assuming that the CKM matrix element $|V_{tb}| \simeq 1$ and neglecting the mass of the bottom quark.

2. Compute the decay width for the process:

$$t \to \phi^+ b\,, \tag{15}$$

where ϕ^+ is the charged Goldstone boson appearing in the decomposition of the Standard Model (SM) scalar doublet (use the top-SM scalar yukawa interactions). Compare to the result obtained in the previous exercise and discuss.

3 Flavours and Discrete symmetries

1. **Charge conjugation**

(a) Under charge conjugation, you have the following transformations:

$$\psi \to C\bar{\psi}^t, \quad A^\mu \to -A^\mu, \quad \phi \to \phi^*, \quad i \to i \tag{16}$$

where ψ is a Dirac Spinor, A^μ is the photon vector field, ϕ is a complex scalar and one can take $C = i\gamma^0\gamma^2$.

We have already mentionned during the lecture that charge conjugation is directly related to complex conjugation. To understand this better take the complex conjugate of the equation:

$$(i\not{D} - m)\psi = 0 \qquad D^\mu = \partial^\mu - iQeA^\mu \tag{17}$$

and after checking that the new gamma matrix $\tilde{\gamma}^\mu = \gamma^2\gamma^{\mu *}\gamma^2$ also satisfy the Clifford algebra, obtain that $\psi^c = C\bar{\psi}^t = -\gamma_2\psi^*$ satisfy the same equation but with an oposite coupling (charge) to the photon field.

(b) Without using the explicit form of the charge conjugation above, check that for the free hamiltonian of a Dirac field to be invariant under charge conjugation, you have to satisfy:

$$-\gamma_\mu^t = C^\dagger \gamma_\mu C\,. \tag{18}$$

Begin with the mass term to first get the $\mu = 0$ result.

(c) Obtain the transformation of $\bar{\psi}\psi, \bar{\psi}\gamma^5\psi, \bar{\psi}\gamma^\mu\psi, \bar{\psi}\gamma^\mu\gamma^5\psi$

2. **Parity transformation** ($\vec{x} \to -\vec{x}$ and $\vec{p} \to -\vec{p}$)

$$\psi \to \gamma_0\psi, \quad A^\mu \to (A^0, -A^i), \quad \phi \to \phi, \quad i \to i \tag{19}$$

Obtain the transformation of $\bar{\psi}\psi, \bar{\psi}\gamma^5\psi, \bar{\psi}\gamma^\mu\psi, \bar{\psi}\gamma^\mu\gamma^5\psi$

3. **Time reversal** ($t \to -t$)

$$\psi \to \gamma_1\gamma_3\psi, \quad A^\mu \to (A^0, -A^i), \quad \phi \to \phi, \quad i \to -i \tag{20}$$

Obtain the transformation of $\bar{\psi}\psi, \bar{\psi}\gamma^5\psi, \bar{\psi}\gamma^\mu\psi, \bar{\psi}\gamma^\mu\gamma^5\psi$

4. Check that weak interactions violate C and P but conserve CP, and determine under which condition the Yukawa interactions are CP invariant.

5. Check that no mixings appear in the SM fermion coupling to the Z (ie no flavour changing neutral currents at tree level)

4 Chiral gauge theories and Anomalies

1. Check that the Noether currents J^μ and J_5^μ associated to

   (a) the vector symmetry transformations $\Psi \to e^{i\alpha}\Psi$

   (b) the chiral symmetry transformations $\Psi \to e^{i\beta\gamma_5}\Psi$

   are conserved for a Dirac field when taking the massless limit.

   Using the Dirac equation in the $m \neq 0$ case compute the expression of $\partial_\mu J^\mu$ and $\partial_\mu J_5^\mu$.

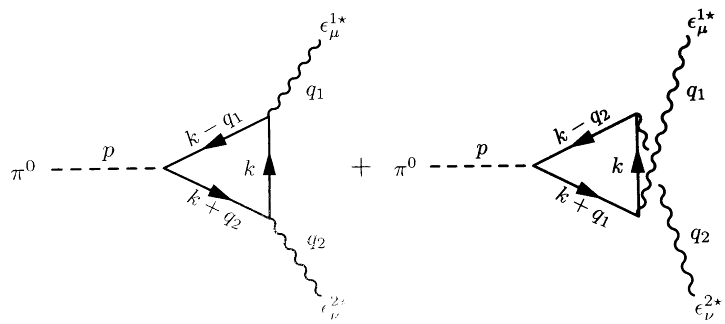

Figure 1: One loop Diagrams involved in $\pi^0 \to \gamma\gamma$

2. When computing the decay width of $\pi^0 \to \gamma\gamma$ involving the diagrams of Fig. 1, with some massive fermion Ψ of mass m running into the loop, one ends up evaluating the following integral over momenta:

$$M^{\mu\nu} = -i \int \frac{dk^4}{(4\pi)^4} \mathrm{Tr} \left(\frac{\gamma^\mu(\not{k} + m)\gamma^\nu(\not{k} + \not{q}_2 + m)\gamma^5(\not{k} - \not{q}_1 + m)}{(k^2 - m^2)((k + q_2)^2 - m^2)((k - q_1)^2 - m^2)} + (\mu \leftrightarrow \nu \& 1 \leftrightarrow 2) \right) \tag{21}$$

   (a) check that the numerator reduces to $4im\epsilon^{\mu\nu\alpha\beta}q_{1\alpha}q_{2\beta}$ using the properties of the traces of gamma matrices.

(b) Making use of the Feynman parameters, use the following identity:

$$\frac{1}{A_1...A_n} = (n-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 ... \int_0^{x_{n-2}} dx_{n-1} \frac{1}{(A_1 x_{n-1} + ... + A_{n-1}(x_1 - x_2) + A_n(1-x_1))^n} \tag{22}$$

to rewrite the denominator of (21) as one single polynomial function of k to the cube.

(c) make a change of coordinate $k \to k' = k'(k, x_i, q_1, q_2))$ such that the latter denominator takes the form of $(k'^2 - \Delta + i\epsilon)^3$ where Δ is independent of k'. You can then use

$$\int \frac{dl^4}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^3} = i \int \frac{dl_E^4}{(2\pi)^4} \frac{1}{(-l_E^2 - \Delta + i\epsilon)^3} = \frac{-i}{32\pi^2 \Delta} \tag{23}$$

resulting from a Wick rotation and l_E the euclidian momentum associated to the 4-momentum l ($l_E^2 = l_0^2 + \vec{l}^2$).

(d) Considering that the 2 outgoing photons are onshell ($q_1^2 = q_2^2 = 0$) and in the limit $m \gg m_\pi$ (e.g. in the case where the fermion in the loop is a proton) check that

$$M^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} \frac{-i}{4\pi^2 m} \tag{24}$$

3. In order to evaluate the chiral anomaly (considering massless fermions), one considers the 3-point correlation function $\langle J^{\alpha 5}(x) J^\mu(y) J^\nu(z) \rangle$ and check if $\partial/\partial x^\alpha \langle J^{\alpha 5}(x) J^\mu(y) J^\nu(z) \rangle$ is zero (same for $\partial/\partial y^\mu$ and $\partial/\partial z^\nu$). Going to momentum space, one ends up evaluateing the same kind of loop diagrams than for the exercise above but without pion or photon external line and the γ_5 insertion is replaced by $\gamma^\alpha \gamma_5$. We will thus deal with

$$M_5^{\alpha\mu\nu} = -i \int \frac{dk^4}{(2\pi)^4} \mathrm{Tr} \left(\frac{\gamma^\mu (\not{k}) \gamma^\nu (\not{k} + \not{q}_2) \gamma^\alpha \gamma^5 (\not{k} - \not{q}_1)}{k^2 (k+q_2)^2 (k-q_1)^2} + (\mu \leftrightarrow \nu \& 1 \leftrightarrow 2) \right) \tag{25}$$

using the same momentum routing as in Fig. 1. Here your goal is to compute $q_{1\mu} M_5^{\alpha\mu\nu}$. The resulting integrand is actually linearly divergent and care has to be taken when shifting the variable over which we are integrating.

In the computation you will encounter the difference between two linearly divergent 4-D integral that differ from one another by a shift a^μ:

$$\Delta^\alpha(a^\mu) = \int \frac{d^4k}{(2\pi)^4} (F^\alpha(k+a) - F^\alpha(k)) \tag{26}$$

making use of Wick rotation and Taylor expansion, assuming that $F^\alpha(k_E) \to A k_E^\alpha / k_E^4$ for $k_E \to \infty$, you get that:

$$\Delta^\alpha(a^\mu) = \frac{i}{32\pi^2 A} a^\alpha \, . \tag{27}$$

Using this result show that $q_{1\mu} M_5^{\alpha\mu\nu} \neq 0$:

$$q_{1\mu} M_5^{\alpha\mu\nu} = \frac{1}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} q_1^\rho q_2^\sigma \, . \tag{28}$$

5 Beyond $SU(3) \times SU(2) \times U(1)$

1. In $SU(5)$, we consider the basis for the fundamental (or vector) representation **5**, or better said its complex conjugate denoted by $\mathbf{5}^*$ or $\bar{\mathbf{5}}$. Notice that here, we will be considering the notation: $\psi_i^* \equiv \psi^i$. [2] Considering that the **5**, transforms as:

$$\psi_i \to U_F{}_i^j \psi_j = \exp(i\alpha^a T_F^a)_i^j \psi_j \tag{29}$$

with $i, j = 1, .., 5$ and $a = 1, .., 24$ with T_F^a satisfying (2) and F referring to the fundamental representation, determine the transformation of the **24** and the **10** in terms of the U matrices of transformation of the fundamental representation. For the latter purpose, use the following hints:

   (a) The **24** is the adjoint representation of SU(5), i.e., the associated generators satisfy $(T_A^b)_{ac} = if^{abc}$, where A refers to the adjoint representation. Using the latter information check that, say a scalar field $\Phi = \Phi^a T_F^a \sim \mathbf{24}$, transforms as $\Phi \to U\Phi U^\dagger$ for U the unitary transformation of (29). use

   (b) The **10** can be obtained from the anti symmetric part of the tensor product $\mathbf{5} \times \mathbf{5}$, or equivalently $\chi_{ij} \sim \frac{1}{2}(\psi_i\psi_j - \psi_j\psi_i)$. As a result show that $\chi \to U\chi U^T$.

Using these results:

   (a) check that (use e.g. $(T^b)_{ac} = if^{abc}$) the covariant derivative takes the form: $D^\mu\Phi = \partial^\mu\Phi - ig[A^\mu, \Phi]$. In addition, taking into account that the adjoint representation has the same quantum numbers as the product of the fundamental represention by the fundamental complex conjugate representation, $\Phi_i^j \sim \psi_i\psi_j^*$, check that the EM charges of each components of $\Phi \sim \mathbf{24}$ can be obtained from $Q(\Phi_i^j) = Q(\psi_i) - Q(\psi_j)$.

   (b) check that the covariant derivative of χ takes the form: $D_\mu\chi = \partial_\mu\chi - ig\{A_\mu, \chi\}$ with $\{A^\mu, \chi\} = A_\mu\chi + \chi A_\mu^T$. In addition, check that the EM charges of each components of $\chi \sim \mathbf{10}$ can be obtained from $Q(\chi_{ij}) = Q(\psi_i) + Q(\psi_j)$.

with $A_\mu = \sum_a A_\mu^a T^a$.

2. In order to obtain the decomposition of tensor products of representations (reducible representations) in $SU(N)$ in terms of irreducible representations Young tableaux appear to be a very efficient tool. For a short introduction, see e.g. `http://pdg.lbl.gov/2014/reviews/rpp2014-rev-young-diagrams.pdf`.

   (a) With the help of Young tableaux, decompose $\mathbf{3} \otimes \mathbf{3}$ and $\mathbf{3} \otimes \bar{\mathbf{3}}$ in SU(3), and $\mathbf{5} \otimes \mathbf{5}$ and $\mathbf{5} \otimes \bar{\mathbf{5}}$ in SU(5). Also decompose $\mathbf{2} \otimes \mathbf{2}$ in SU(2). Anything special about the latter?

   (b) The quintuplet of SU(5) can be decomposed as a direct sum of multiplets of SU(3)×SU(2): $\mathbf{5} = (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$. Compute $\mathbf{5} \otimes \mathbf{5}$ and $\mathbf{5} \otimes \bar{\mathbf{5}}$ in this decomposition and find the decomposition for the **10**, **15** and **24**. Identify the SM gauge bosons in the adjoint representation.

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[2] In the SU(5) GUT model that we will consider, we will take $\psi_i^* \equiv \psi^i = (d_1^c, d_2^c, d_3^c, e^-, -\nu_e)_L^T$.

6 Aspects of Neutrino physics

1. Considering a neutrino a flavour α as a combination of mass eigenstates: $|\nu_\alpha\rangle = \sum_i U_{\alpha i}|\nu_i\rangle$

   (a) Check that, after having prapagated along some distance L, the probability for ν_α to ν_β transition is given by :

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \sum_{ij} J^{ij}_{\alpha\beta} \exp\left(-i\Delta m^2_{ij} L/(2E)\right) \\ &\text{with} \quad J^{ij}_{\alpha\beta} = U^*_{\beta j} U_{\alpha j} U_{\beta i} U^*_{\alpha i} \\ &\text{and} \quad m^2_{ij} = m^2_j - m^2_i \end{aligned} \tag{30}$$

   (b) Using the unitarity of the mixing matrix U, show that:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}[J^{ij}_{\alpha\beta}] \sin^2\left(\Delta m^2_{ij} L/(4E)\right) \\ &\quad +2\sum_{i>j} \mathrm{Im}[J^{ij}_{\alpha\beta}] \sin\left(\Delta m^2_{ij} L/(2E)\right) \end{aligned} \tag{31}$$

   (c) Using the above result and making use of $\Delta m^2_{12} \ll \Delta m^2_{13}$, the transition probability in the 3 family case effectively reduces to a 2 family problem in several cases. Show for example that considering $E/L \sim \Delta m^2_{13}$ the disappearance probability $P(\nu_e \to \nu_e)$ is at leading order a function of θ_{13} and Δm_{13}.

2. Majorana versus Dirac:

   (a) For a Majorana fermion: $\mathcal{L} = \bar{\psi}_L i\partial\!\!\!/ \psi_L - \frac{m}{2}(\bar{\psi}_L \psi^c_L + h.c.)$,

      i. check that the above Majorana mass term is equivalent to $-\frac{m}{2}(\psi^T_L C \psi_L + h.c.)$

      ii. check that for $\psi_M = \psi_L + \psi^c_L$, one has: $\mathcal{L} = \frac{1}{2}\bar{\psi}_M(i\partial\!\!\!/ - m)\psi_M$

   (b) Also for Dirac fermion $\psi = \psi_L + \psi_R$, defining $\psi_{M1} = \psi_L + \psi^c_L$ and $\psi_{M2} = \psi_R + \psi^c_R$, show that $m\bar{\psi}\psi = \frac{m}{2}\bar{\psi}_{M1}\psi_{M2} + \frac{m}{2}\bar{\psi}_{M2}\psi_{M1}$

3. A $n \times n$ Majorana mass matrix, contains how many physical phases ?

References

[1] Michael E. Peskin and Daniel V. Schroeder. An Introduction to quantum field theory. 1995.

[2] Beringer et al. (Particle Data Group). Review of particle physics. *Phys. Rev. D*, 86:010001, Jul 2012.