GGI LECTURES 2025
ON
DARK MATTER COSMOLOGY.

From a selection of DM production mechanisms to a selection of cosmology probes.

PART II.

LAURA LOPEZ HONOREZ (ULB)
LLOPEZHO@ULB.BE.

IV. Boltzmann equations.

The Question that arises is how DM is produced given a set of properties. In order to answer this question, we typically need to handel evolution beyond equilibrium. Our main tool are Boltzmann equations that drive the DM phase space distribution function $f_\chi(p^\mu, x^\mu)$ that counts the number of particles in the phase space element $d^3x \; d^3p$ where x an p refer to position and momentum. Here in particular we follow the convention:

$$\begin{cases} n_\chi = \displaystyle\int \frac{d^3p}{(2\pi)^3} \, f_\chi \\ \rho_\chi = \displaystyle\int \frac{d^3p}{(2\pi)^3} \, E(p) \, f_\chi \end{cases}$$

where χ refers to DM (scalar, fermion, ...). careful, here we absorb the χ number of dof into f_χ.

The Boltzmann equation can be written as:

$$\boxed{L[f] = C[f]}$$

influenced by cosmology

influenced by Particle phys.

IV 1. The Liouville operator

- $$L[f] = \frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu}.$$

↳ parameter that monotonically increase along the path

when particles move freely, we expect them to follow geodesic eqs.

$$p^\mu = \frac{dx^\mu}{d\lambda} = (E, \vec{p}) \quad \xrightarrow{\text{geod eq}} \quad \frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

$$\Rightarrow \boxed{L[f] = p^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial f}{\partial p^\alpha}} \quad (*)$$

this part of the Boltzmann equation is essentially influenced by the cosmology through $\Gamma^\alpha_{\beta\gamma}$.

NB: $L[f]$ above is a covariant relativistic generalization of the non relativistic (NR) Liouville operator that you can find in astro-oriented text books

$$L_{NR} \equiv \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \vec{x}}{\partial t}\frac{\partial}{\partial \vec{x}} + \frac{\partial \vec{v}}{\partial t}\frac{\partial}{\partial \vec{v}}$$

$$= \frac{p^\mu}{p^0}\frac{\partial}{\partial x^\mu} + \frac{\partial p^i}{\partial x^0}\frac{\partial}{\partial p^i} \sim \frac{L}{p^0}$$

- Considering FRW metric, with spatially homogeneous and isotropic PSD: $f(x^\mu, p^\mu) = f(t, |\bar{p}|) \equiv f(t, E)$, the Liouville operator, at zeroth order in perturbations*, reduces to:

$$L[f] = E\left(\frac{\partial}{\partial t} - H\frac{|\bar{p}|^2}{E}\frac{\partial}{\partial E}\right) f(t, E)$$

NB $L[f] \neq f_{;0}$.

in particular above we have

$$L[f] = E\frac{df}{dt}$$

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* we mostly leave the details of cosmological perturbation evolution to reference cosmo books: Dodelson, Baumann, etc

IV 2. C[f] the collision term

C[f] is the collision term that captures the particle physics interactions, of your species X.

On very general grounds, one should consider both <u>elastic</u> and <u>inelastic</u> processes:

$$C[f] = C_{el}[f] + C_{inel}[f]$$

Below, we provide order of magnitude estimates for a DM <u>TOY MODEL</u> with X coupling to weak gauge bosons.

$\leftrightarrow C_{el}[f]$

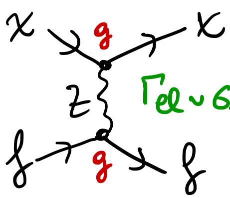
IV 2.1. Elastic scatterings and Thermal equilibrium

A <u>Rough Estimate of Kinetic decoupling</u>.

See 0104173, 0612238.

- for a DM "toy model":

$$\chi f \to \chi f$$



$\Gamma_{el} \sim G_{el}\, n_f \propto G_F$

elastic processes

↳ drives momentum exchanges.

- Temperature of "last scattering" (T_{ls}).

$$\tau_{col} = \Gamma_{el}^{-1}(T_{ls}) = H^{-1}(T_{ls}) = t_H$$

with $\tau_{col} \equiv$ "<u>collision" time</u>.

- Even before the time of last scattering (local) thermal equilibrium can be lost!

One defines the relaxation time, τ_{rel}, as the time = time necessary to return to (local) thermal equilibrium.

→ it is estimated as the time necessary to change $p_x = p$ significantly.

- Let us evaluate τ_{rel} for a non relativistic specie x ($m_x \ll T$) which interacts through weak interactions.

⇝ typical of WIMP DM, see below

* in each scatterings on light species, you can hope to change p as

$$\frac{\Delta p}{p} \sim \left(\frac{T}{m_x}\right)^{1/2} \ll 1.$$

$\Delta p \sim E_f \sim T$

$E_x^{kin} = \frac{p^2}{2m} \sim T \Rightarrow p \sim (Tm)^{1/2}$

* For random walk in momentum space, the total change in moment. after N_{coll} is expected to scale as:

$$\frac{\Delta p_{tot}}{p} = N_{coll}^{1/2} \frac{\Delta p}{p} \sim \left(N_{coll} \frac{T}{m_x}\right)^{1/2}.$$

* To change significantly the momentum you thus need:

$$\boxed{N_{coll} \sim \frac{m_\chi}{T}} \gg 1$$

$$\Rightarrow \quad \tau_{rel} \sim N_{coll}\, \tau_{col} \sim \frac{N_{coll}}{\Gamma_{el}}$$

* Thermal equilibrium of χ is thus lost $\equiv$ Kinetic Decoupling (KD) when:

N_{coll} for $\Delta p_{tot} \sim p$

$$\boxed{\tau_{rel} \sim \frac{N_{coll}}{\Gamma_{el}} \sim H^{-1}(T_{KD}) \sim t_H}$$

in particular in the context of our WIMP toy model

$$\begin{cases} \bullet\ \Gamma_{el} \sim \sigma_{el} \times n_f \quad (n_f \to T^3) \\ \bullet\ H \sim \dfrac{T^2}{M_p} \end{cases} \qquad \sigma_{el} \sim g^4 \frac{T^2}{m_\chi^4}$$

N_{coll}

$$\Rightarrow \frac{m_\chi/T}{g^4 T^5/m_\chi^4} \sim \left(\frac{T^2}{M_p}\right)^{-1}$$

see e.g. 0104173 for a concrete example.

$$\Rightarrow T_{KD}^4 \sim \frac{m_\chi^5}{g^4 M_p} \sim \frac{m_\chi^5}{G_F^2\, m_N^4\, M_p}$$

Take $m_\chi \sim 100$ GeV and $G_F \sim 10^{-5}$ GeV^{-2}

$$\Rightarrow \quad T_{KD} \sim 10^{-7/4}\,\text{GeV} \sim 20\,\text{MeV}.$$

$$\sim \alpha^{-1/2}\, m_\chi^{5/4}$$

$$\alpha \sim \frac{g^2}{4\pi}$$

$$\Rightarrow \quad T_{KD} \ll m_\chi \,!$$

Notice that this is a very rough estimate but the order of magnitude is recovered when computing the full set of Boltzmann equations.

NB For weakly interacting particles (electrons, "vanilla WIMP", ...) KD usually happens well after chemical decoupling (see below) so that one usually directly make use of Fermi-Dirac, Bose-Einstein or Boltzmann (non relativistic) momentum dependent distribution functions.

NB Last scattering can be roughly evaluated as $\Gamma_{el}(T_{ls}) = H(T_{ls})$

$$\rightarrow \quad T_{ls} \sim \left(\frac{m_\chi^4}{G_F^2 m_W^4}\,\frac{1}{M_P}\right)^{1/3} \sim \text{MeV} \lesssim T_{KD}.$$

B. Kinetic decoupling : temperature evolution equations.

- When particles are suspected to have a departure from thermal equilibrium with the heat bath, one should follow the "unintegrated" Boltzmann equations $\frac{d f(p,t)}{dt}$ which can be computationally expensive and tricky.
(integro partial differential equation, see e.g. 1706.07433)

- It has been shown though that when chemical and kinetic decoupling are intertwined it might be enough to follow
→ see 2.2: number density evolution
the first and second momenta only of distribution function. Here, we define χ temperature as:

$$\boxed{T_\chi = \frac{1}{n_\chi}\left\langle \frac{1}{3}\frac{\bar{p}^2}{E}\right\rangle = \frac{1}{n_\chi}\int \frac{d^3p}{(2\pi)^3}\frac{\bar{p}^2}{3E} f_\chi(\bar{p},t)}$$

NB: ∘ for non relativistic particles

$$T_\chi \simeq \frac{1}{n_\chi}\frac{2}{3}\left\langle \frac{\bar{p}^2}{2m_\chi}\right\rangle$$

∘ another dimensionless variable is:

$$y_\chi = \frac{m_\chi}{s^{2/3}}\left\langle \frac{\bar{p}^2}{3E}\right\rangle = \frac{m_\chi}{s^{2/3}}\frac{1}{n_\chi}\int \frac{d^3p}{(2\pi)^3}\frac{\bar{p}^2}{3E} f_\chi(\bar{p},t)$$

see e.g. 0612238, 1706.07433, 2103.01944.
1603.04884.

- The T_X evolution equations can be obtained from the Boltzmann equations for $f_X(\bar{p},t)$: [0903.0189, 1603.04884, etc]

$$\overbrace{\int \frac{d^3p}{(2\pi)^3} \cdot \frac{L[f]}{E} \frac{p^2}{3E}}^{①} \cdot = \overbrace{\int \frac{d^3p}{(2\pi)^3} \frac{C[f]}{E} \frac{p^2}{3E}}^{②} .$$

* On the one hand, the Liouville part:

$$① = \partial_t (n_X T_X) - H \int \frac{d^3p}{(2\pi)^3} \left(\frac{|\bar{p}|^2}{E} \partial_E(f) \right) \frac{p^2}{3E}$$

$$= n_X (\partial_t T_X - \underbrace{3H}\, T_X) - H \frac{4\pi}{(2\pi)^3} \underbrace{\int \frac{dp}{3} \cdot \cancel{p^2} \frac{p^4}{E^{\cancel{2}}} \overbrace{\frac{\cancel{E}}{\cancel{p}} \partial_p f}^{\partial_E f_p}}.$$

$\partial_t n_X + 3H n_X = 0$
neglecting inelastic coll!

by parts:

$$-\frac{1}{3} \int dp\, f\, \partial_p \underbrace{\left(\frac{p^5}{E} \right)}$$

$$\frac{5p^4}{E} - \frac{p}{E} \frac{p^5}{E^2}$$

$$\rightarrow ② = n_X \left(\partial_t T_X + 2 H T_X - H \left\langle \frac{1}{3} \frac{p^4}{E^3} \right\rangle \right)$$

$\begin{cases} \text{For non relativ. DM: we neglect the last term as } p^2/E^2 \ll 1 \\ \text{For relativistic DM: } \frac{1}{3}\langle p^4/E^3\rangle = \frac{1}{3}\langle p\rangle = T_X . \end{cases}$

As a result, the Liouville contribution reduces to:

$$① = \begin{cases} m_X\,(\partial_t T_X + 2H\,T_X) & \text{for N.R. DM} \\ m_X\,(\partial_t T_X + H\,T_X) & \text{for relativ DM.} \end{cases}$$

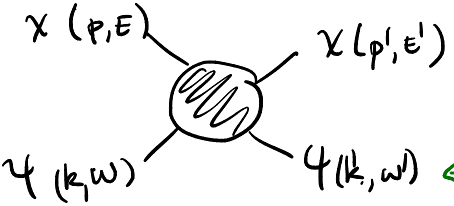
Considering the case of kinetic decoupling i.e. when the $\int C_{el} \to 0$ (elastic collisions can not maintain KE) we would have $② = 0$, i.e.

$$\boxed{T_X \propto \begin{cases} 1/a^2 & \text{for KD N.R. DM.} \\ 1/a & \text{for KD relativistic DM.} \end{cases}}$$

$\to$ ⚠ $\neq T \propto h_{eff}^{-1/3}\, a^{-1}$ for coupled DM!

* the collision term for elastic processes

$$\chi\psi \leftrightarrow \chi\psi \equiv$$



← some SM specie

$$C_{el}[f] = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2\omega}\int\frac{d^3k'}{(2\pi)^3 2\omega'}\int\frac{d^3p'}{(2\pi)^3 2E'}\,(2\pi)^4\,\delta^4(p+k-p'-k')$$

DM assumed not affected by BE-PB effects.

$$|M|^2_{\chi\psi\leftrightarrow\chi\psi}\Big(\big(1\pm f_\psi(\omega)\big)f_\psi(\omega')f_\chi(E') - \big(1\pm f_\psi(\omega')\big)f_\psi(\omega)f_\chi(E)\Big)$$

$\int\frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}\times\frac{1}{E}$ ↓ after some lengthy calculation

for $T \lesssim m_\chi$ and $t \ll m^2$ ← mandelstam variable!

$$② = \frac{3}{2}\underbrace{\gamma(T)}\,m_\chi(T - T_\chi)\left(1 + \mathcal{O}\left(\frac{p^2}{E^2}\right)\right)$$

momentum transfer rate
$\sim T/m \times$ scattering rate
(due to random walk in momentum space)

$$\Rightarrow\ \boxed{\dot{T}_\chi + 2HT_\chi = \frac{3}{2}\gamma\,(T - T_\chi)\,.}$$

for N.R. DM

$$\rightarrow T_\chi = \begin{cases} T & \text{when } \gamma \text{ drives the evol } (T \gg T_{KD}) \\ \dfrac{cst}{a^2} & \text{when } H \text{ drives the evol } (T \ll T_{KD}) \end{cases}$$

NB: keeping track of g_*, using $x = m_X/T$

$$\rightarrow \quad \frac{d\ln y}{d\ln x} = \frac{\gamma(T)}{H}\left(1 - \frac{y_{eq}}{y}\right)$$

$$* \; \gamma(T) = \frac{1}{48\pi^3 m_X^3}\frac{1}{T}\int d\omega\, f_\psi(1 \pm f_\psi) k^4 \langle |M|^2\rangle_t$$

$$\text{w/ } \langle |M|^2\rangle_t = \frac{1}{8k^4}\int_{-4k^2}^{0} dt\,(-t)\,|M|^2$$

$$= 16\pi\, m_X^2\, \sigma_{el}$$

transfer Xsection

$$\sim c_n \frac{\omega^n}{m_X^n} \; ⊛ .$$

see e.g. [1603.04884]

- $f_\psi(1 \pm f_\psi) = -T\,\partial_\omega f_\psi$

for ψ thermal distrib.

$$\rightarrow \quad \gamma(T) = \frac{1}{48\pi^3 m_X^3}\int d\omega\, f_\psi\, \partial_\omega\left(k^4 \langle |M|^2\rangle_t\right)$$

$$= \overbrace{g_\psi\, T^3}^{n_\psi}\, \sigma_{el}(\omega = T) \times \underbrace{\frac{T}{m_X}}_{\text{Need for significant } \Delta p.} \times \text{cst}$$

$$\text{cst} = \frac{\Gamma(4+n)\,\zeta(4+n)}{3\pi} \times \begin{cases} 1 & \psi = b \\ (1 - 2^{-n-3}) & \psi = f \end{cases}$$
⊛.

we find that the right momentum transfer rate shall scale as
$$n_4 \cdot \sigma_{el} \, N_{cole}$$
as used in the approx method of section A.

NB$_2$: One could also assume that DM is not in K.E with the SM bath but yet with another bath. in the later case, similar evolution equations can be introduced where $\xi = \dfrac{T'}{T}$

with $\begin{cases} T' = \text{dark bath temp} \\ T \quad \text{SM.} \end{cases}$

shall enter (see e.g. 1603.04884)

C. DM free streaming

As a side note, notice that particles free-stream (FS) from the moment they are kinetically decoupled leaving an exponential cut on the power spectrum due to collision-less damping (≡ free streaming)

→ this assumes χ with $SU(2)_L$ interactions, non relativistic at the time of KD. $M_{FS} = \frac{4\pi}{3} \lambda_{FS}^3 \, \rho_\chi$

- For the WIMP toy model considered you get *
$M_{FS} = 10^{-6} M_\odot$ for $\begin{cases} 100 \text{ GeV DM} \\ T_{KD} \sim 30 \text{ MeV} \end{cases}$

- electrons get chemical decoupling at $T \sim$ MeV but get kinetically decoupled at $z \sim 10^2$ only (from which $T_{gaz} \propto 1/a^2$ instead of $T \propto 1/a$)

$\Rightarrow$ they are still in kinetic equilibrium with photons at recombination ($z \sim 10^3$) as expected from the observation of a blackbody spectrum (≡ thermal equ.) for the CMB spectrum up to deviations $\sim 10^{-5}$!
(see details in 0612238)

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* a small galaxy of ~ 100 kpc size $\Leftrightarrow M \sim 10^9 M_\odot$ which roughly corresponds to the smallest scales tested by Ly-α forest data.

- Note that DM weakly interacting with the thermal bath but chemically & kinetically decoupling while relativistic can get strong constraints from F.S. or ΔN_{eff}. Same comment for e.g non-thermal DM (FI, SuperWIMP) considered below for more details see Cosmology constraints sections.

- Let us emphasize that free-streaming is one possible source of damping. Another one is collisional damping (Silk Damping) due to e.g. interactions of massive species with lighter ones, as is the case of baryons or DM scattering with neutrinos or (dark) photons

see also eg: 0012504, 0410591, 0903.0189
1205.5809, 1603.04884

IV 2.2 Description of particles in thermal equil with a heat bath.

- When following a thermal equilibrium evolution early on, one has:

$$\boxed{f_X(p,t) = \frac{g_X}{\exp\left(\frac{E-\mu}{T} \pm 1\right)}}$$

where $\mp$ stand for $\begin{cases} \text{BE} \\ \text{FD} \end{cases}$ species.

and μ is the chemical potential and g_X count the number of internal degrees of freedom of X.

- When particles are in thermal equilibrium it is enough to follow the first momentum of the distribution function

number density.

$$\begin{cases} n_X = \displaystyle\int \frac{d^3p}{(2\pi)^3}\, f_X(p,t) \\ \\ \rho_X = \displaystyle\int \frac{d^3p}{(2\pi)^3}\, E\, f_X(p,t) \end{cases}$$

energy density

* In the $\left\{\begin{array}{l}\text{N.R.}\\ \text{relativistic}\end{array}\right.$ limit ($T \gg m_\chi, \mu$)

$\zeta(3)$ → Riemann function $\zeta(3) \simeq 1.2$

$$\begin{cases} n_\chi = g_\chi^n \dfrac{\zeta(3)}{\pi^2} T^3 \\ \rho_\chi = g_\chi^\rho \dfrac{\pi^2}{30} T^4 \end{cases}$$

where $g_\chi^n = \left\{\begin{array}{l}1\\ 3/4\end{array}\right. g_\chi$ and $g_\chi^\rho = \left\{\begin{array}{l}1\\ 7/8\end{array}\right. g_\chi$ for $\left\{\begin{array}{l}\text{bosons}\\ \text{fermions}\end{array}\right.$

* In the non-relativistic limit ($m_\chi \gg T$)

$$f_\chi \sim \exp\left(\frac{E-\mu}{T}\right) \equiv \text{Boltzmann distribution}$$

$$\begin{cases} n_\chi = g_\chi \left(\dfrac{m_\chi T}{2\pi}\right)^{3/2} \exp\left(-(m_\chi-\mu)/T\right) \\ \rho_\chi = m_\chi n_\chi \end{cases}$$

- Notice that for most of the cases we are interested in the DM production will happen in a radiation dominated (RD) era:

$$\boxed{H = \frac{T^2}{M_0(T)} \quad \text{where} \quad M_0(T) = M_p\sqrt{\frac{45}{4\pi^3 g_*(T)}} \simeq 1.66\,\frac{T^2}{g_*(T) M_p}}$$

[Radiation dominated (RD) era]*

where $g_*(T)$ denotes the number of relativistic dof contributing to radiation density at temperature T: $\rho_R = \frac{\pi^2}{30} g_* T^4$

- In addition the entropy density can be defined as: $\boxed{s = \frac{2\pi^2}{45} h_{eff} T^3}$ for a collection of relativ. species.

On general grounds g_* and h_{eff} differ when ν decouple from the heat bath so that $T_\nu(T) < T$

$$\begin{cases} g_*(T) = \sum_i g_i^f \left(\frac{T_i}{T}\right)^4 \\ h_{eff}(T) = \sum_i g_i^f \left(\frac{T_i}{T}\right)^3 \end{cases} ; \quad g_i^f = g_i \times \begin{cases} 1 & \text{bosons} \\ 7/8 & \text{fermions} \end{cases}$$

[also denoted $g_{*s}(T)$]

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* see e.g. 2102.06221, 2408.08350, etc for DM production in a matter dominated (M.D) era.

- we can also introduce the dimension-less number density.

$$\boxed{Y = \frac{n}{s}}$$

Y is often refered to the "comoving number density" but carefull $Y \neq n(t)/a^3$!!

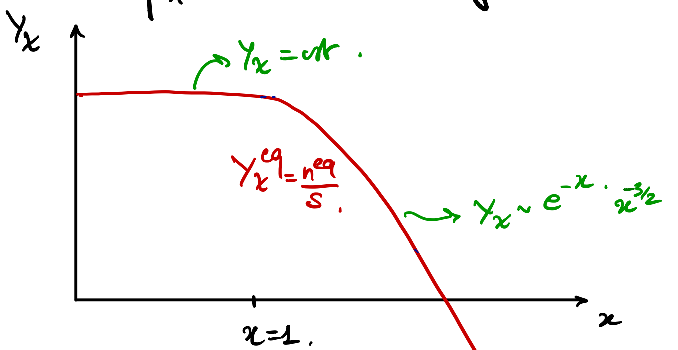
- It is convenient to introduce a dimensionless time variable $\sim m_{ref}/T$

$$x = \frac{m_{ref}}{T}.$$

where m_{ref} is some reference mass. In particular for $m_{ref} = m_\chi$

$$\boxed{x = \frac{m_\chi}{T}}$$

Making use of this convenient variable, the equilibrium comoving number density at $\mu=0$, commonly refered to as "Y_{eq}" takes the form:



NB: In cases where f_{eq} is well approximated by a Maxwell Boltzmann statistic : $f_i^{eq} = g_i \exp(-E/T)$ one obtains :

$$n_i^{eq}(x) \overset{\text{assuming M.B. statistic.}}{\simeq} \frac{g_i}{2\pi^2} \frac{m_i^3}{x} \underbrace{K_2[x]}_{\text{Modified Bessel funct° of the 2d kind.}} = \begin{cases} \dfrac{g_i}{\pi^2} \dfrac{m_i^3}{x^3} & \text{for } x \ll 1 \\[2ex] g_i \dfrac{m_i^3}{(2\pi x)^{3/2}} e^{-x} & \text{for } x \gg 1 \end{cases}$$

with $x = m_i/T$

* $K_2(x) = x \int_1^\infty (t^2-1)^{1/2}\, t \exp(-tx)\, dt$

$t = E_i/T$.

* Notice that in the limit $x \ll 1$, i.e. the relativistic limit, MB statistic is wrong by a factor $\begin{cases} \zeta(3) & \text{for bosons} \\ \frac{3}{4}\zeta(3) & \text{for fermion} \end{cases}$

compared to the use of the F.D. and B.E. statistics

- From 1st law of Thermodynamics + Conservation equations, you get:

$$\boxed{\frac{d(sa^3)}{dt} = 0.}$$ Entropy conservation.

For particles in equ.; in absence of particles injection (e.g. reheating BH evap, etc...)

$$\Rightarrow \quad \boxed{T \propto h_{eff}^{-1/3} \; a^{-1}}$$

$\Rightarrow$ iff. particles are in eq. and $h_{eff} = cst$

$$\Rightarrow T \propto 1/a$$

Now, particles that are decoupled from the heat bath (kinetic decoupling see above.), i.e., when we can neglect their interactions with the heat bath, they do not participate anymore to entropy transfers and

particle temperature $T_X(t)$ may differ from $T(t)$! heat bath temperature

actually $T_X(t) \propto \begin{cases} 1/a & \text{for relativ. species} \\ 1/a^2 & \text{for N.R. species} \end{cases}$

⚠ no h_{eff} dependence!

• $g_*(T) = h_{eff}(T)$ at high T

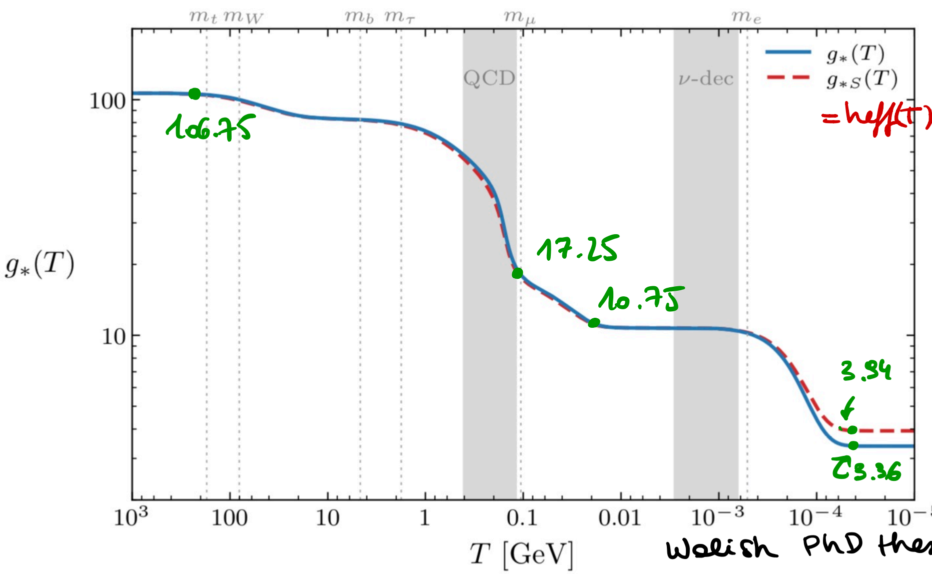
for SM only $\begin{cases} g_*(T) \simeq 100 \text{ for } T > T_{EW} \\ h_{eff}(T) \end{cases}$

while today:

$$\begin{cases} g_*(T_0) = \overset{2}{g_\gamma^0} + \overset{7/8 \times 2}{g_\nu^0} \times N_{eff}^\nu \times \left(\frac{T_\nu^0}{T_0}\right)^4 = 3.36 \,. \\ h_{eff}(T_0) = g_\gamma^0 + g_\nu^0 \times N_{eff}^\nu \times \left(\frac{T_\nu^0}{T^0}\right)^3 = 3.91 \end{cases}$$

$$\left.\frac{T_\nu}{T}\right|_{T<m_e} = \left(\frac{h_{eff}(T_{CD}^\nu)}{h_{eff}(T^0)}\right)^{1/3} = \left(\frac{4}{11}\right)^{1/3}$$

by entropy conserv. at $T<m_e$ when e^+e^- heat the γ bath.



Walish PhD thesis

NB : • actually the γ_L decoupling was not completely instantaneous and actually not fully complete at the time where $e^+e^- \to \gamma\gamma$ and their distribution is not exactly a F.D. distribution. As a result, in ⊛, we should take this into account:

$$N_{eff}^{\nu} \neq N_{\nu} = 3$$
$$= 3.046$$

see {1606.06986, 1812.05605}

• More generally, one could account for other unknown relativistic species. Those are usually accounted for as ΔN_{eff}, i.e. the number of extra relativistic degrees of freedom counted as:

$$\boxed{\Delta N_{eff}(T) = \frac{\rho_{rel}^{extra}(T)}{\rho_{rel}^{\nu}(T)/N_{eff}^{\nu}} = \frac{\sum_{i=extra} g_i \left(T_i/T\right)^4}{2 \times \frac{7}{8}\left(\frac{T_\nu}{T}\right)^4}}$$

those are currently constrained to be

$$\Delta N_{eff}(T_{CMB}) < 0{,}28 \text{ at } 95\%\ CL \text{ (Planck)}$$

and to a similar number by BBN, while Euclid + future CMB data

$$\Delta N_{eff} < 0{,}063$$

[2405.06047]

- Also, assuming that at the time of matter-radiation equality $a = a_{eq}^{MR}$ we can consider that all ν are relativistic we have:

$$\Omega_r(a_{eq}^{MR}) = \Omega_\gamma(a_{eq}^{MR}) + \Omega_\nu^{rel}(a_{eq}^{MR})$$

$$\Rightarrow \Omega_\nu^{rel}(a_{eq}^{MR}) = \frac{\rho_\nu^{rel}}{\rho_c} = \frac{N_{eff}^\nu}{\rho_c} \frac{7}{8} \frac{\pi^2}{30} g_\nu T_\nu^4(a_{eq}^{MR})$$

$$T_{dec}^\nu > T_{MReq} \Rightarrow T_\nu^4(a_{eq}^{MR}) = \left(\frac{4}{11}\right)^{4/3} T_\gamma^4(a_{eq}^{MR})$$

$$\Rightarrow \Omega_\nu^{rel}(a_{eq}^{MR}) = \frac{N_{eff}^\nu}{\rho_c} \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \times g_\nu \frac{\rho_\gamma}{g_\gamma}$$

$$= \underbrace{N_{eff}^\nu \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}}_{0.69} \Omega_\gamma(a_{eq}^{MR})$$

$g_\nu = g_\gamma = 2$

$$\Rightarrow \Omega_r(a_{eq}^{MR}) \simeq 1.69\, \Omega_\gamma(a_{eq}^{MR}).$$

$$\Rightarrow \frac{\Omega_r(a_{eq}^{MR})}{\Omega_m(a_{eq}^{MR})} = 1 = \frac{1.69\, \Omega_{\gamma,0}}{a_{eq}^{MR}\, \Omega_{m,0}}$$

$$\Rightarrow \boxed{a_{eq}^{MR} = 2.8\, 10^{-4}}$$

accounts for $N_{eff}^\nu = 3.046$ rel ν at M-R eq.

$$\Leftrightarrow z_{eq}^{MR} = 3506$$

$C_{inel}[f]$

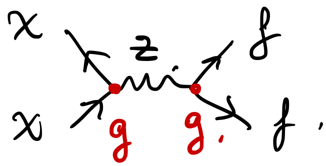
2.3. Inelastic Scatterings and Chemical decoupling

In addition to potentially fast momentum exchanges with the heat bath through elastic scatterings (see sec. 2.1), the DM might have inelastic interactions with particles of the heat bath such as, e.g. annihilations.

For our "toy" model, we would have e.g.:

$$\chi\bar{\chi} \to f\bar{f}$$

inelastic processes.



$\Gamma_{ann} \propto \sigma_{ann} \propto G_F$

Chemical decoupling (CD) happens when these inelastic processes, or particle number density changing processes, become slower than the expansion rate of the universe: ie

$$\boxed{\Gamma_{inel} \sim H(T_{CD})}$$

Notice that for CD of N.R DM of our toy model, you expect:

$$T_{CD} \sim \frac{m_\chi}{25} > T_{KD} \sim \text{few MeV}$$ for 100 GeV particle.

Let us refer to in $\leftrightarrow$ fin $+ X$ as all the possible processes giving rise to a variation of your dark matter number density:

$$\boxed{\frac{1}{E_X} C_{inel}[f] = \frac{1}{2E_X}\int \prod_\alpha \left(\frac{d^3p_\alpha}{(2\pi)^3 2E_\alpha}\right)(2\pi)^4\delta^4(P_{fin}+p_X-P_{in}) \times\left[|M|^2_{in\to fin+X}\, f_{in}\,\underbrace{\left(1\pm\frac{f_X}{g_X}\right)}_{\text{BE enhancement / Fermi suppression}}\left(1\pm\frac{f_{fin}}{g_{fin}}\right) - |M|^2_{fin+X\to in}\, f_{fin}.f_X\left(1\pm\frac{f_{in}}{g_{in}}\right)\right]}$$

where the above notations correspond to:

- the α index is <u>not</u> a Lorentz index here but runs over all species in the in and fin state
- $P_{fin} = \sum_{\alpha\ni fin} p_\alpha$; $P_{in} = \sum_{\alpha\ni in} p_\alpha$.

  ie P_{fin} is the sum of the 4-momenta of particles in the final state.
- $\left(1\pm\frac{f_X}{g_X}\right)$ is a $\begin{cases}\text{Pauli-Blocking (PB,}-)\\ \text{Bose-Einstein enhancement (BE,}+)\end{cases}$ factors for $\begin{cases}\text{fermionic DM}\\ \text{bosonic}\end{cases}$

while $\left(1\pm\frac{f_{in}}{g_{in}}\right)$, $\left(1\pm\frac{f_{out}}{g_{out}}\right)$, correspond to a <u>product</u> of PB, BE factors for particles in the in or fin states.

- f_{in}, f_{fin} are the product of phase space distributions for particles in the in/fin state.
- $|M|^2_{i\to f}$ are the transition matrix element squared averaged over in states and summed over fin states.

Assuming 1/ CP invariance so that the amplitude squared summed of in and fin

$$|M|^2 = g_{in} |M|^2_{in \to out + X} = g_{out} g_X |M|^2_{out + X \to in}$$

and that 2) we consider cases without Bose condensation or fermi degeneracy, i.e.
$$\left(1 \pm f_i/g_i\right) \simeq 1 .$$

The collision term, which we will refer to as $C[f]$ from now on, reduces to a simpler:

$$\boxed{\begin{aligned} \frac{1}{E_X} C[f_X] = \frac{1}{2E_X} \int \prod_\alpha \left(\frac{d^3 p_\alpha}{(2\pi)^3 2E_\alpha} \right) (2\pi)^4 \delta^4 (P_{fin} + p_X - P_{in}) \\ \times |M|^2 \left[\frac{f_{in}}{g_{in}} - \frac{f_{fin} f_X}{g_{fin}\, g_X} \right] \end{aligned}}$$

For the rest of the lecture, I will assume that 1/ and 2/ hold and I will use the above version of the Boltzmann equ.

- as we will focus on the number density evolution let us rewrite the Liouville operator in terms of n_χ

$$\frac{1}{E} L[f_\chi] = \partial_t f_\chi - H \frac{|\vec{p}|^2}{E} \partial_E f_\chi .$$

$$\begin{cases} \bullet \displaystyle\int \frac{d^3p}{(2\pi)^3} \partial_t f_\chi = \partial_t n_\chi \\ \bullet \displaystyle\frac{1}{(2\pi)^3} \underbrace{\int d^3p \frac{p^2}{E} \partial_E f_\chi}_{4\pi \int_0^\infty dp\, p^2 \frac{p^2}{E} \underbrace{\partial_E f_\chi}_{\frac{E}{p} \partial_p f_\chi}} = -3 n_\chi \end{cases}$$

$$\frac{dE}{dp} = \frac{p}{E} \longrightarrow \frac{E}{p} \partial_p f_\chi$$

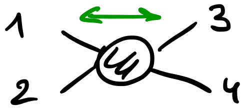
$$\underbrace{4\pi \int_0^\infty dp\, p^3 \partial_p f_\chi}_{\underset{p.p.}{=} 4\pi \left(-\int_0^\infty dp\, 3 p^2 f_\chi + \left[p^3 f_\chi \right]_0^\infty \right)}$$

$$\left[p^3 f_\chi \right]_0^\infty \rightarrow 0$$

$$\Longrightarrow \boxed{\frac{1}{(2\pi)^3} \int d^3p \frac{1}{E} L[f_\chi] = \partial_t n_\chi + 3 H n_\chi}$$

$\rightsquigarrow$ when there are no interactions $\partial_t (n_\chi a^3) = 0$ ✓

- Among the particle number changing processes we will mainly work with $2 \leftrightarrow 2$ processes



In the latter case for $1 = \chi$

$$\frac{1}{E_1} C[f_1] = \frac{1}{2E_\chi}\int \prod_\alpha \left(\frac{d^3 p_\alpha}{(2\pi)^3 2E_\alpha}\right)(2\pi)^4\delta^4(P_{fin}+p_\chi-P_{in})$$

$$\times |M|^2 \left[f_{in}/g_{in} - f_{fin} f_\chi/(g_{fin} g_\chi)\right]$$

$$= \frac{1}{2E_\chi}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\,\frac{d^3p_3}{(2\pi)^3 2E_3}\,\frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$(2\pi)^4\delta^4(p_3+p_4-(p_1+p_2))$$

$$|M|^2\left(\frac{f_3 f_4}{g_3 g_4}-\frac{f_1 f_2}{g_1 g_2}\right)$$

* We will also be particularly interested in cases in which $T \ll E-\mu$

$$\Rightarrow \quad f_{FD/BE} \longrightarrow f_i(E) = g_i \exp\left(\frac{-(E-\mu_i)}{T}\right)$$

$$\Rightarrow \quad \boxed{\frac{f_i(p_i,t)}{f_i^{eq}(p_i,t)} = \exp\left(\frac{-\mu_i}{T}\right) = \frac{n_i(t)}{n_i^{eq}(t)}}$$

in addition energy conservation

$$\Rightarrow \quad \exp(-(E_1+E_2)/T) = \exp(-(E_3+E_4)/T)$$

$$\Rightarrow \frac{f_1^{eq} f_2^{eq}}{g_1 g_2} = \frac{f_3^{eq} f_4^{eq}}{g_3 g_4}$$

In the latter case:

$$\frac{1}{E_1} C[f_1] = \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$\times (2\pi)^4 \delta^4(p_3+p_4-(p_1+p_2))$$

$$\times \frac{|M|^2}{g_1 g_2} f_1^{eq} f_2^{eq} \left[\frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}} \right].$$

* Also, remember from your QFT lectures: average over initial dof

$$\sigma_{12\to 34} = \underbrace{\frac{1}{g_1} \frac{1}{g_2}}_{\text{average over initial dof}} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^4 2E_4}$$

$$(2\pi)^4 \delta^4(p_3+p_4-(p_1+p_2)) \frac{|M|^2}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}$$

$\sum_{dof} M^\dagger M$

denoting $\sigma_{12\to 34} \equiv \sigma_{12}$ and let us define:

4-momentum

$$v_{ij} = \frac{\sqrt{(p_i\cdot p_j)^2 - m_i^2 m_j^2}}{E_i E_j}$$

$\equiv$ Møller velocity.

$$\Rightarrow \frac{1}{E_1} C[f_1] = \left[\frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}} \right] .$$
$$\int \frac{d^3 p_2}{(2\pi)^3} \, \sigma_{12} v_{12} \, f_1^{eq} f_2^{eq}$$

Let us define the thermally averaged cross section

$$\langle \sigma v \rangle_{12} = \frac{1}{n_1^{eq} n_2^{eq}} \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \sigma_{12} v_{12} \, f_1^{eq} f_2^{eq}$$

and remembering that

$$\int \frac{d^3 p_1}{(2\pi)^3} \left(\frac{1}{E_1} L[f_1] = \frac{1}{E_1} C[f_1] \right)$$

$$\boxed{\partial_t n_1 + 3H n_1 = \langle \sigma v \rangle_{12} n_1^{eq} n_2^{eq} \left[\frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}} \right] .}$$

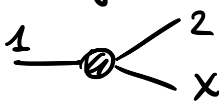
terms showing competition between H and $\Gamma_{inel} \sim n_2 \langle \sigma v \rangle_{12}$

- when $H \gg \Gamma_{inel} \leadsto \partial_t n_1 + 3H n_1 = 0 \equiv \frac{\partial_t (n_1 a^3)}{a^3}$
  $\Rightarrow$ the comoving number density is conserved
- when $H \ll \Gamma_{inel}$ it enforces chemical equ:
  $$\left[\frac{n_3 n_4}{n_3^{eq} n_4^{eq}} - \frac{n_1 n_2}{n_1^{eq} n_2^{eq}} \right] = 0 \Leftrightarrow e^{(\mu_3 + \mu_4)} = e^{(\mu_1 + \mu_2)}$$
  $$\Rightarrow \mu_3 + \mu_4 = \mu_1 + \mu_2$$

- **NB** $\langle \sigma v_{12} \rangle \times n_1^{eq} n_2^{eq}$ has no explicit dependence in $g_1 g_2$

$$\sim \frac{1}{n_1^{eq} n_2^{eq}} \underbrace{\sigma_{12} v_{12}}_{\sim \frac{1}{g_1 g_2}} \underbrace{f_1^{eq} f_2^{eq}}_{\sim g_1 g_2.}$$

- Considering processes of the type:



assuming $m_1 \gg T$ but at kin eq. you can show that

$$\boxed{\partial_t n_\chi + 3H n_\chi = \langle \Gamma_{1\to 2X} \rangle \, n_1^{eq} \left[\frac{n_1}{n_1^{eq}} - \frac{n_2 \, n_X}{n_2^{eq} \, n_X^{eq}} \right]}$$

where $\langle \Gamma_{1\to 2X} \rangle = \Gamma_{1\to 2X} \dfrac{K_1(T/m_1)}{K_2(T/m_1)}$

and $K_1(x) = x \int_1^\infty (t^2-1)^{1/2} \exp(-tx) \, dt$

$t = E_i/T$.

V the FREEZE-OUT MECHANISM.

driven by (co-) annihilations.

Let us consider first, the "textbook" case of a DM particle that is thermal $\equiv$ kinetic equilibrium with the heat bath at early time*.

In the latter case, since early times, Y_x will follow Y_x^{eq} up until when chemical decoupling will happen. At that point, if no other number changing processes light on, we expect Y_x freeze. = FREEZE-OUT.

In order to evaluate the chemical decoupling temperature, T_{CD}, one uses $\boxed{\Gamma_{ann} = H(T_{CD})}$

If T_{CD} happens in radiation dominated era (ie $T_{CD} > T_{eq}$ and no early matter dominated epoch)

$$H = \frac{T^2}{M_0(T)} \quad \text{where} \quad M_0(T) \sim \frac{M_P}{g_*^{1/2}}$$

---

* Let us emphasize that the heat bath is usually the SM bath, however, the dark sector could very well be thermally decoupled with $\underset{DS}{T'} \neq \underset{VS}{T}$ see e.g. 2105.01263

In these lectures, we refer to T as the SM heat bath temperature.

We have multiple examples of particles in kinetic and chemical equ. at early times that decouple at later times.

The most obvious one is the SM neutrino, ν_L.

→ we know that they have a non zero hypercharge and isospin combined to give $Q_\nu = 0$

→ ν_L interacts with weak interactions with the SM heat bath at early times

Let's assume for a minute that we do not know about laboratory and cosmology constraints and let's try to evaluate for which mass m_ν the SM neutrinos could account for all the DM.

In the next sections, we will go step by step. Let us however flash the final result for the ν_L abundance shown on p.8.

1. Hot relics ≡ Freeze-out while relativistic

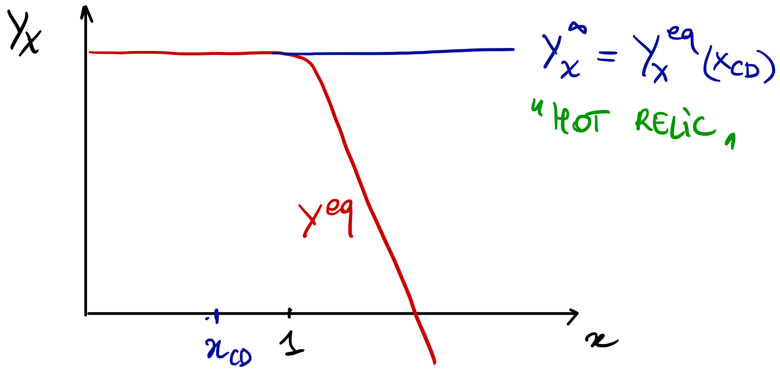
- In the case where our χ particle would be sufficiently coupled to the heat bath at early time ($T \gg m_\chi$) <u>and</u> if the particle interactions are such that $T_{CD} > m_\chi$ the particle will decouple while still being relativistic

$$\Rightarrow \quad Y_\chi^\infty = Y_\chi^{eq}(x_{CD}) = \frac{n_\chi(T_{CD})}{s(T_{CD})}$$

$$= 0{,}278 \, \frac{g_\chi^n}{h_{eff}(T_{CD})}$$

$$\Rightarrow \boxed{\Omega_\chi^0 h^2 \underset{\not{o}}{=} \frac{Y_\chi^\infty}{\rho_c^0/h^2} s_0 m_\chi = 0{,}12 \left(\frac{g_\chi^n \, m_\chi}{GeV}\right) \frac{h_{eff}^0}{h_{eff}(T_{CD})}}$$

FO HOT RELIC



- This type of hot relics is exactly the case of SM neutrinos.

In the case of a 2 dof fermion DM *

i.e. $g_\chi^n = \frac{3}{4}\, 2$

in the form of DM interacting through weak interactions:

neutrinos decoupling Temp

i.e. $\Gamma_{ann} \sim \langle \sigma v \rangle n_\chi \sim G_F^2 T^5$ for $m_\chi \ll T_{D\nu}$

$\Rightarrow \Gamma_{ann}(T_{D\nu}) = H(T_{D\nu}) \Rightarrow T_{D\nu} \sim O(\text{MeV})$

$\Rightarrow h_{eff}(T_{D\nu}) \simeq g_\gamma^s + g_e^s + \overset{N_\nu}{3}\, g_\nu^s = 10.75.$

$$\Rightarrow \boxed{\Omega_\chi h^2 = \left(\frac{g_\chi}{2}\right) \frac{m_\chi}{92\,\text{eV}} \qquad \text{HOT RELIC EWly interacting fermion}}$$

A well known bound on thermal DM is obtained imposing that the HOT RELIC should satisfy $\Omega_\chi h^2 < 1 \Rightarrow m_\chi < 92\,\text{eV}$.
= COSWIK AND Mc Clelland bound

For $\Omega_\chi h^2 < 0{,}12 \Rightarrow m_\chi = \sum_\nu m_\nu \lesssim 10\,\text{eV}$.

---

* SM neutrinos are either Majorana, in which case $g_\nu^M = 2$ or Dirac, but the $\nu_R = \nu_{sterile}$ does not couple to Z boson, so that again $g_\nu^D = 2$

- One can be more general being agnostic about the nature of the hot relic in which case (*) can always be rewritten as:

$$\Omega_X h^2 = \left(\frac{m_X}{92\,eV}\right)\left(\frac{h_{eff}(T_{D\nu})}{h_{eff}(T_D)}\right)$$

$\Omega_X h^2 = 0{,}12 \rightarrow$ $\boxed{h_{eff}(T_D) = 974\left(\frac{m_X}{keV}\right)}$

assuming a 2 dof fermion decoupling at temperature T_D, while for neutrinos we saw that $h_{eff}(T_{D\nu}) = 10.75$.

This implies in particular that in order to have a viable hot relic of a few keV mass (usually refered to as thermal warm dark matter) one would need ~1000 relativistic dof at the time of its decoupling. Keeping in mind that the SM only has ~100 of those, this requieres a non negligible BSM content in the most straigthforward applications.(*)

(*) One can extend possibilities considering DM coupled to a hidden sector with $T' \neq T$ see e.g. 2105.0126

Considering that both chemical and kinetic decoupling happen at the same time, one can compare the DM temperature T_X today to the one of the SM to which it was assumed to be coupled at $T > T_D$:

while X is relativistic
$$\begin{cases} T \propto (h_{eff}(T))^{-1/3}\, a^{-1}. \\ T_X \propto a^{-1} \end{cases}$$

3.91

$$\frac{T_X(T_0)}{T_0} = \left(\frac{h_{eff}(T_0)}{h_{eff}(T_D)} \right)^{1/3}.$$

$$\rightarrow \boxed{ \left.\frac{T_X}{T}\right|_{t_0} = 0{,}16 \left(\frac{m_X}{\text{keV}} \right)^{-1/3} \left(\frac{\Omega_X h^2}{0{,}12} \right)^{1/3}. }$$

this will be usefull for later purposes.

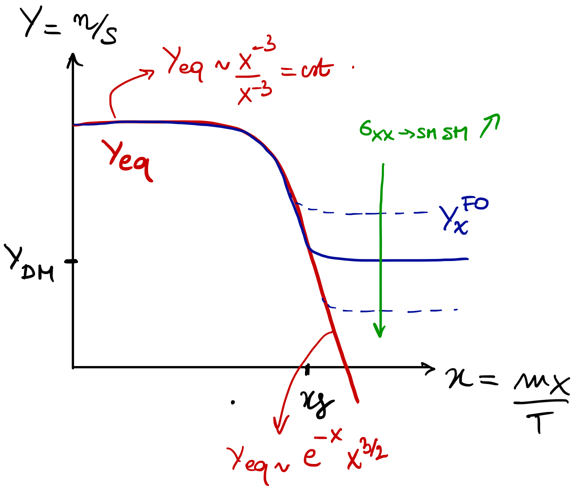
2. Cold Relics = FO. while Non relativistic

This is the general case of the so-called WIMP, weakly interacting massive particles.

Let us emphasize the "weakly interacting" is not = "$SU(2)_L$ interacting" but more generally "interacting with $g \sim O(g_{SU(2)_L})$"

For cold relics, it is assumed that DM decouples when N.R. ie while $Y_\chi \sim e^{-x} x^{3/2}$

In the latter case the DM number density evolution goes as follows:

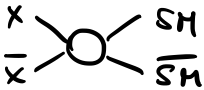


2.1 Vanilla WIMP : approximate results

Here, we just rapidly evaluate the DM relic abundance in the instantaneous freeze-out approximation.
A more precise evaluation can be found in. Gondolo - Gelmini '91.

Consider the following process $\chi\bar{\chi} \leftrightarrow SM\,\overline{SM}$ for keeping χ in equilibrium :

$$\Gamma_A = n_\chi \langle \sigma_A . v \rangle$$



$\langle\ \rangle$ denotes thermal average

• We first estimate the time (or temperature) at which the freeze-out occurs :

$n_\chi = n_\chi^{N.R.}$

$$\Gamma_A = n_\chi \langle \sigma_A v \rangle \hat{=} H(T_{CD})$$

$$\Leftrightarrow \langle \sigma_A . v \rangle . g_\chi \frac{m_\chi^3}{(2\pi x_f)^{3/2}} e^{-x_f} \simeq \frac{T_{CD}^2}{M_0(T_{CD})}$$

$x_f = \frac{m_\chi}{T_{CD}}$

$$= \frac{m_\chi^2}{x_f^2 M_0}$$

take $\sqrt{g_*} \sim 10$. $g_\chi \sim 2$

$$\Leftrightarrow \langle \sigma_A v \rangle m_\chi . M_p \sqrt{x_f} \sim \exp x_f .$$

$$\Rightarrow x_f \simeq \frac{1}{2} \ln x_f + \ln(\langle \sigma_A v \rangle m_\chi M_p)$$

Considering $m_X \gg m_{SM}$ you can have
$\langle \sigma_A v \rangle \sim \frac{1}{m_X^2} = cst$

Also considering $m_X \ll M_p$ you can approximate:

$$\boxed{x_f \cong \ln\left(\langle \sigma_A v \rangle m_X \cdot M_p\right) \quad \text{and} \quad T_{CD} \doteq \frac{m_X}{x_f} < m_X.}$$

Considering $\begin{matrix} X \\ X \end{matrix} \rangle \sim Z \langle \begin{matrix} SM \\ \overline{SM} \end{matrix}$: $m_X \sim \mathcal{O}(100\,\text{GeV}) \to x_f \sim 25.$

- We can now estimate the dependence of Ω_X in $\langle \sigma_A v \rangle$ using $\Gamma_A = H(T_{CD})$.

$$\Gamma_A = H(T_{CD}) \Rightarrow n(T_{CD}) = \frac{1}{\langle \sigma_A v \rangle} \frac{T_{CD}^2}{M_p}$$

$$\& \; n(T_0) = n(T_{CD}) \frac{a_{CD}^3}{a_0^3} \sim n(T_{CD}) \frac{T_{CD}^{-3}}{T_0^{-3}}$$

$$\Rightarrow \Omega_X h^2 \sim n_X(T_0) m_X \propto \frac{m_X}{\langle \sigma_A v \rangle M_p T_{CD}}.$$

$$\text{i.e. } \Omega_X h^2 \propto \frac{x_f}{\langle \sigma_A v \rangle M_p}.$$

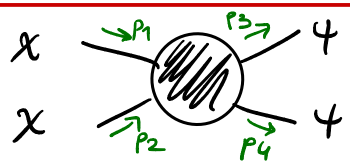
2.2. Vanilla WIMP : More details

Going back to the Boltzmann equations one can recover the exact result for $\Omega_\chi h^2$.

Let us work as in the previous set-up, where DM interacts with SM dof without caring much about potential dark sector or visible sector mediators. I refer to this case as Vanilla WIMP and calculation details can be found in Gondolo Gelmini '91

Here, we are going to assume that elastic $\chi\, SM \leftrightarrow \chi SM$ and inelastic (annihilations) $\chi\chi \leftrightarrow SM\, SM$ are happening fast enough to ensure that DM is in kinetic equ (we can use (FD, BE, MB distributions) and chemical equ in the early universe. Assuming that $\chi SM \leftrightarrow \chi SM$ decouple after $\chi\chi \leftrightarrow SM\, SM$, we can focus on the Boltzmann equ involving $\chi\chi \leftrightarrow SM SM$ only, i.e.:

$$\boxed{\frac{dn_1}{dt} + 3Hn_1 = \langle \sigma_{12} v_{12} \rangle \left(n_1^{eq} n_2^{eq} - n_1 n_2 \right).}$$



This equation can easily be rewritten in terms of the dimensionless variables:

$$\boxed{\begin{array}{l} \dfrac{dY_1}{dx} = \dfrac{s\langle \sigma_{12} v_{12} \rangle}{\bar{H} x} \left(Y_1 Y_2 - Y_1^{eq} Y_2^{eq} \right) \\[2ex] \text{where} \quad \bar{H} = H \left(1 + \frac{1}{3} \dfrac{d\ln h_{eff}}{d\ln T} \right)^{-1} \end{array}}$$

$\bar{H}$ comes from the fact that we assume that entropy is conserved

$$\frac{d(a^3 s)}{dt} = 0 \quad \text{and} \quad s \propto h_{eff} T^3$$

$$\Rightarrow \boxed{\frac{d\ln T}{d\ln t} = -\bar{H}}$$

Keeping in mind that for DM self annihilation $m_i = m_j$, we have

$$\boxed{\frac{dn}{dt} + 3Hn = \langle \sigma_A v \rangle \left(n_{eq}^2 - n^2 \right)}$$

VANILLA WIMP.

this is valid for both $X = \bar{X}$ and $X \neq \bar{X}$

- One can also define the relative velocity v_{rel} as:

$$v_{rel} = \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}{s} \neq v_{mol}$$

In the center of mass frame $v_{rel} = 4\frac{|\vec{p}_{1cm}|}{\sqrt{s}} = 2v_1^{cm}$

- Notice though that for $\underline{m_i = m_j \Rightarrow s = (2E_i)^2}$

$\Rightarrow$ $\underline{v_{mol} = v_{rel}}$

In the latter framework, in the N.R. lim

$$\sigma_A v_{lab} = a + b\,\overset{\sim v_1^{cm}}{\varepsilon^2} + \dots \qquad \varepsilon = \frac{s-4m^2}{4m^2}$$

a = s-wave term , b = p-wave term , ...

$$\rightsquigarrow \langle\sigma_A v\rangle = g_1 g_2 (n_1^{eq} n_2^{eq})^{-1} \int d^3p_1\, d^3p_2\, \sigma v_{lab}\, e^{-(E_1+E_2)/T}$$

$$\underset{N.R.}{=} \sqrt{\frac{x^3}{4\pi}} \int_0^\infty dv\, v^2\, e^{-\frac{x v^2}{4}}\, \sigma_A v_{lab}$$

and $\boxed{\langle\sigma_A v\rangle = a + \frac{6b}{x} + \dots}$

It can be shown that for $x > x_{CD} = x_f$

$$\boxed{\Omega_\chi h^2 = 0{,}12 \left(\frac{2.2\,10^{-26}\ cm^3/s}{a + 3b/x_f + \dots}\right)\sqrt{\frac{80}{g_*}}\ \frac{x_f}{23}}$$

and we have assumed $g_* = cst$ and $h_{eff} = cst$

- Let us take the example of DM $\equiv \nu_L$. For $m_\nu >$ MeV and $m_\nu < m_Z \Rightarrow \sigma_A v \propto G_F^2 m_\nu^2$

$$\Rightarrow \Omega_\nu h^2 \sim \frac{1}{m_\nu^2}$$

and going through the details of the cross-section, one would get the right DM abundance for $m_\nu = 6$ GeV for a COLD relic

- Unitarity limits the annihilation cross-section reads as.

$$\boxed{\sigma_A v < \frac{4\pi\,(2J+1)}{m_\chi^2\, v}}$$

Griest & Kamionkowski '90. [GK]

for s-wave annihilation taking

$$v|_{FO} \sim \sqrt{\langle v_1^2 \rangle} = \sqrt{\frac{6}{x_f}} \quad \text{with } x_f = 25$$

and $\sigma_A v = 2.2\,10^{-26}\, cm^3/s \simeq 2\,10^{-9}\, GeV^{-2}$

$\Rightarrow$ $m_\chi \sim 100$ TeV. saturates the unitary bound.

$\rightsquigarrow$ One expects thus at high mass

$$\Omega_\nu h^2 \sim \left.\frac{1}{\sigma_A v}\right|_{GK} \sim m_\nu^2$$

see also 2105.01263 for a generalisation with a HS at T' on VS at T.

$\rightarrow$ higher mass unitary bound if $T' < T$

.2.3 Co-annihilations. (Griest & Seckel '91)

from WIMP to FIMP.

"weak" interactions — "feeble" interactions

Let us now assume that the DM χ is not the only Z_2 odd particle. Let us introduce $\chi_i = 1 \ldots N$; $m_i > m_j$ for $i > j$ and $\chi_1 = DM = \chi$.

If these dark sector (Z_2 odd) particles are close in mass with χ, they will affect the final DM abundance.

In order to determine the DM relic abundance, one should a priori take into account all

- self- / co- annihilation processes with e.g.

$$\sigma_{ij} = \sigma(\chi_i \chi_j \to SM\, SM')$$

- conversion processes and elastic scatterings, decays and inverse decays, with e.g.

$$\begin{cases} \sigma_{i\to j} = \sigma(\chi_i\, SM \to \chi_j\, SM') \\ \Gamma_{i\to j} = \Gamma(\chi_i \to \chi_j\, SM) \end{cases}$$

- The generic form of the N. Boltzmann equations would be:

$$\frac{dn_i}{dt} + 3H n_i = -\sum_{j=1} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{eq} n_j^{eq} \right)$$

$$- \sum_{j \neq i, SM} n_{SM}^{eq} \Big(\langle \sigma_{i \to j} v_{iSM} \rangle (n_i - n_i^{eq}) - \langle \sigma_{j \to i} v_{jSM} \rangle (n_j - n_j^{eq}) \Big)$$

$$- \sum_{j \neq i} \Big(\langle \Gamma_{i \to j} \rangle (n_i - n_j^{eq}) - \langle \Gamma_{j \to i} \rangle (n_j - n_j^{eq}) \Big)$$

- Side note

Notice that we can rewrite the above eqs in a more compact form using the reaction rates for $2 \leftrightarrow 2$ and $1 \leftrightarrow 2$ processes as:

$$\begin{cases} \gamma_{ij \to kl} = \iint d\phi_i \, d\phi_j \, f_i^{eq} f_j^{eq} \int d\phi_k \, d\phi_l \\ \qquad \times (2\pi)^4 \delta^4(p_i + p_j - p_k - p_l) \left| M_{ij \to kl} \right|^2 \\ \qquad = n_i^{eq} n_j^{eq} \langle \sigma_{ij} v_{ij} \rangle \\ \gamma_{k \to ij} = \int d\phi_k \, f_k^{eq} \\ \qquad \times (2\pi)^4 \delta^4(p_i + p_j - p_k) \left| M_{k \to ij} \right|^2 \\ \qquad = n_k^{eq} \langle \Gamma_{k \to ij} \rangle \end{cases}$$

where $d\phi_i = \dfrac{d^3 p_i}{(2\pi)^3 2E_i}$; i, j, k, l denote both SM and DS particles.

$$\langle \Gamma_{k \to ij} \rangle = \Gamma_{k \to ij} \, \overbrace{K_1(x) / K_2(x)}^{\langle 1/\gamma \rangle} = \text{thermally averaged decay rate.}$$

Assuming no $\cancel{CP}$, we get for dimensionless variables

$$\boxed{\begin{aligned} H x s \frac{dY_i}{dx} = &-\sum_{jk} \gamma_{ij\to kl} \left(\frac{Y_i Y_j}{Y_i^{eq} Y_j^{eq}} - \frac{Y_k Y_l}{Y_k^{eq} Y_l^{eq}} \right) \\ &- \sum_{jk} \gamma_{k\to ij} \left(\frac{Y_i Y_j}{Y_i^{eq} Y_j^{eq}} - \frac{Y_k}{Y_k^{eq}} \right). \end{aligned}}$$

• ① Assuming that short after DM FO, all $\{X_i\}_{i\neq 1}$ decay to DM $= X_1$, one can write for

$$n = \sum_{i=1}^{N} n_i$$

$$\frac{dn}{dt} + 3Hn = \sum_{ij=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{eq} n_j^{eq} \right),$$

② Here we assume that all the relevant particles are in thermal and chemical equi in the early universe. In particular, assuming $x_i \leftrightarrow x_j$ conversion processes happen fast enough $\left(\Gamma_{x_i \leftrightarrow x_j} > H(x_f) \right)$ so as to consider them in chemical equ.

ie

$$\mu_i \simeq \mu_j \quad \Rightarrow \quad \frac{n_i}{n_i^{eq}} = e^{-\mu_i/T} = e^{-\mu_j/T} = \frac{n_j}{n_j^{eq}}$$

and

$$\frac{n_i}{n_i^{eq}} = \frac{n}{n^{eq}} \quad \text{with } n_{eq} \simeq \sum_i g_i \int \frac{d^3 p_i}{(2\pi)^3} e^{-E_i/T}$$

This allows to simplify even more the Boltzmann eq., which reduces to:

$$\boxed{\begin{aligned} &\dot{n} + 3Hn = -\langle \sigma_{eff} v \rangle \left(n^2 - n_{eq}^2 \right) \\ &\text{with } \langle \sigma_{eff} v \rangle = \sum_{ij} \langle \sigma_{ij} v_{ij} \rangle \frac{n_i^{eq} n_j^{eq}}{n^{eq}\, n^{eq}} \end{aligned}}$$

In particular, when considering the DM χ_1 and a Z_2 odd partner χ_2, we have 3 type of possibilities:

$$\Omega_\chi h^2 \propto \frac{1}{\langle \sigma v_{eff} \rangle} \propto \begin{cases} \dfrac{1}{\langle \sigma_{11} v \rangle} & \sigma_{11} \gg \sigma_{12}, \sigma_{22} \quad \equiv \text{DM annihil. F.O.} \\[2ex] \dfrac{1}{\langle \sigma_{12} v \rangle \exp(-\Delta x)} & \sigma_{12} \gg \sigma_{11}, \sigma_{22} \quad \equiv \text{co-annihilation F.O.} \\[2ex] \dfrac{1}{\langle \sigma_{22} v \rangle \exp(-2\Delta x)} & \sigma_{22} \gg \sigma_{11}, \sigma_{12} \quad \equiv \text{mediator ann. F.O.} \end{cases}$$

The relative mass difference $\Delta = \dfrac{m_1 - m_2}{m_1}$ should not be too small for co-annihilations to play a role.

NB Here we see in particular that in the case in which χ_1 coupling to SM and χ_j would be smaller than the ones of χ_j to e.g. SM <u>but</u> still large enough to keep thermal an chemical equ., one could very well have the DM relic abundance fully driven by χ_j annihilation if $\Delta_j = \frac{m_j - m_1}{m_1}$ is small enough.

In this extreme case, that I refer to as "<u>mediator annihilation FO</u>", one has for the extra DS particle χ_2:

$$\langle \sigma_{eff} v \rangle \Big|_{\substack{med \\ ann}} = \langle \sigma_{22} v_{22} \rangle \, \frac{g_2^2}{g_{eff}} \, (1+\Delta_i)^3 \, e^{-2x\Delta_2}$$

$$g_{eff} = \sum_i g_i \, (1+\Delta_i)^{3/2} \exp(-x\Delta_i)$$

ie. the crossection setting the relic abundance is "independent" of $\chi_1 \leftrightarrow SM$ interactions

3. DM-, co-, mediator-annihilation FO, one illustrative case:

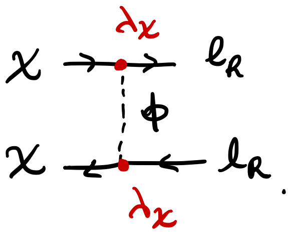
Let us illustrate the transition in DM–SM coupling strength from "vanilla WIMP" to the case of mediator annihil FO. in the case of the leptophilic scenario introduced before.

$$\mathcal{L} \supset \lambda_\chi \bar{\chi} \, l_R \, \phi + h.c.$$

DM, self conjugate Majorana fermion

↳ EM charged scalar.

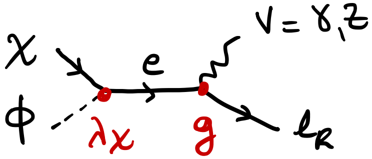
DM annihilation:



$$\sigma_{11} \propto \lambda_\chi^4$$

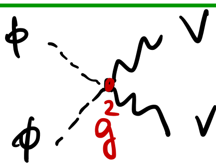
NB: see extra contribs on p 47.

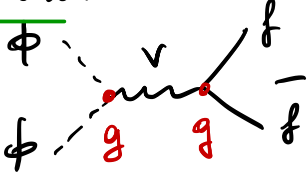
Co-annihilations, eg:



$$\sigma_{12} \propto \lambda_\chi^2 g^2$$

Mediator annihilations:

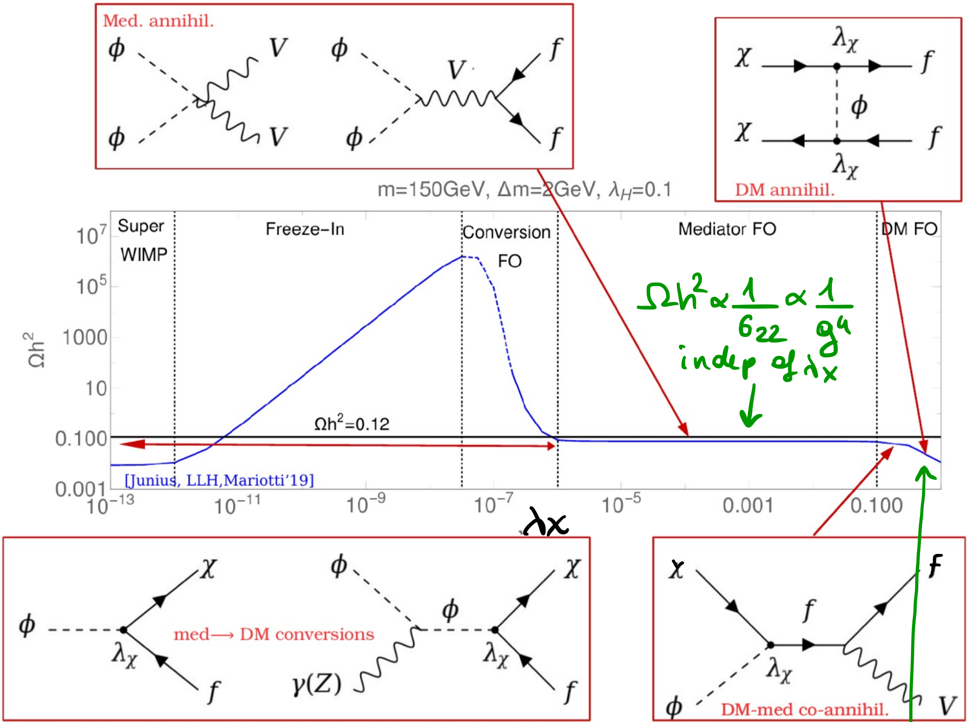




$$\sigma_{22} \propto g^4$$

Dedending on the $\Delta_{12} = \frac{m_\phi - m_\chi}{m_\chi}$ and the relative strength / # of channels involved in the annihilation / co-annihilation channels the DM relative abundance of 0.12

can be obtained for $\lambda_\chi > 10^{-6}$ though the FO mechanism assuming <u>both</u> chemical <u>and</u> kinetic equ between DM-med and SM. $\rightarrow$ not really a "WIMP"

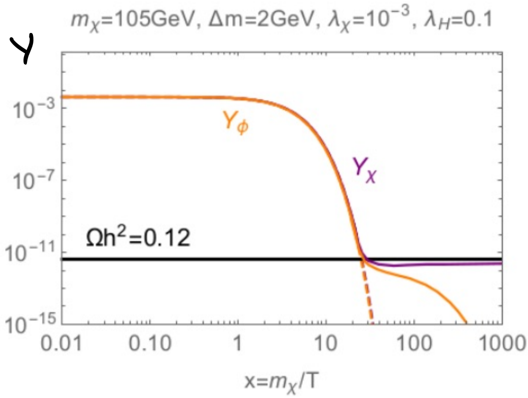


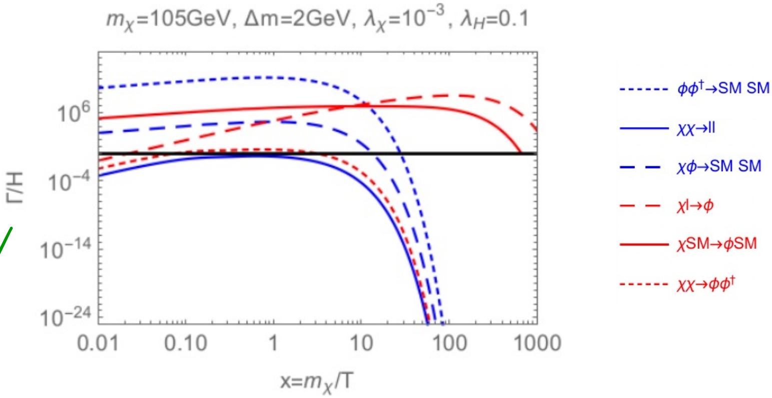
$\Omega h^2 \propto \frac{1}{\sigma_{22}} \propto \frac{1}{g^4}$ indep of λ_χ

$\Omega h^2 \propto \frac{1}{\sigma_{11}} \propto \frac{1}{\lambda^4}$

Mediator annihilation FO in the leptophilic scenario (credit: Sam Junius)

Y

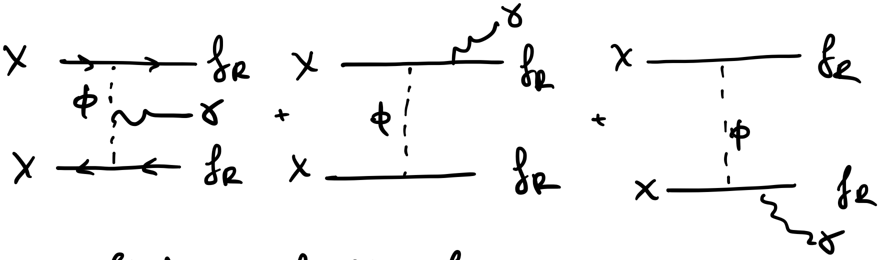




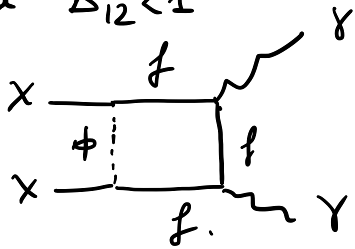
$$\Gamma = \frac{\gamma_{ij}}{n_i^{eq}}$$

* an interesting note on the "t-channel" models is that you can have nice line-like features from.

- Virtual-internal-Bremsstrahlung.



which is large when
$m_f \ll m_X$ and $\Delta_{12} < 1$

- loops like eg. 

⇝ relevant for indirect detection searches but also relic abundance if $f_R = q_R$. especially when X is a self conjugate scalar field.

See eg. 1307.6480, 1511.04452, 1503.01500, ...

- Also if f is colored, non perturbative effects due to multiple gauge boson exchanges for the colored mediator ϕ have to be taken into account! (Sommerfeld effect see e.g. 2308.01336. Bound state ...)

VI FEEBLY INTERACTING MASSIVE PARTICLES

The case of feebly interacting massive particles (FIMPS) is usually directly associated to the Freeze-in mechanism of production.

Here I will refer to FIMPS as particles that interact with the SM or the mediator of interactions with coupling much more feeble than weak interactions $\rightsquigarrow$ out of chem/kin equ from the start.

Within this framework I will describe 3 production mechanism:

1| the Freeze-in (Fi)
2| the superWIMP mechanism (SW)
3| the $\left\{\begin{array}{l}\text{conversion}\\ \text{coscattering}\end{array}\right.$ driven Freeze out. (conv F.O.)

- Going from FI to SW mechanisms, smaller couplings are involved.
- We can also go from
  Fi $\to$ convFO $\to$ co-annihilations FO $\to$ FO,
  by increasing the couplings <u>but</u> smalle couplings between the DM and a dark sector partner is needed.

Here we will work in a framework where :

- $\mathbb{Z}_2$ odd particles : $\left\{ \begin{array}{l} B = \text{both particle in thermal } \underline{\text{and}} \text{ chemical eq at early time} \\ \chi = \text{DM} \end{array} \right.$

$m_B > m_\chi$

- Production happens in a radiation dominated era i.e. $H(T_{prod}) \sim \frac{T^2}{M_p}$.

in the case of e.g FI :

- Notice that you can also produce DM directly from SM in models such as
$\left\{ \begin{array}{l} \text{dark photon mediated interactions} \\ \text{Higgs portal} \end{array} \right.$
<u>or</u> with the mother particle out-of equilibrium (sequential FI).
See e.g. 1908.09864, 2005.06294, 0911.1120.

- DM production in a modified early cosmology can have very interesting impact for FIMP detection at colliders and the interplay with cosmology
<u>see</u> e.g. 2102.06221 for an early MD era

1. Freeze-in

Here we will deepen the case of FI from a mother particle decay into SM particle(s) and the DM : $B \to \chi$.

1.1. Rule of thumb

One can guess the typical dependence of Y_χ in terms of Γ, m_B and M_P.

Indeed the amount of DM particles at a given time for a production rate R should be

$$Y_\chi \sim R \cdot t .$$

Lorentz factor
time dilation
$\Delta' t = \Delta t \cdot \gamma$

$\gamma = \frac{E_B}{m_B} \sim \frac{T}{m_B} = x^{-1}$

Considering $\bullet\ t \sim \frac{1}{H(T)} \sim \frac{M_P}{T^2}$

with

$\boxed{x = m_B/T}$ $\bullet\ R = \langle \Gamma_{B\to\chi} \rangle \propto \Gamma_{B\to\chi} \cdot x$

$\hookrightarrow x \ll 1$
$\Leftrightarrow m_B \ll T$

NB $x = m_B/T$ is correcting the rest frame B decay rate in the thermal bath.

$$\Rightarrow Y_\chi \propto \Gamma_{B\to\chi} \frac{m_B}{T} \frac{M_P}{T^2} \propto \frac{\Gamma_{B\to\chi} M_P x^3}{m_B^2}$$

$\rightsquigarrow$ the DM production is more efficient at low T $\to$ IR dominated process for FI through decays

Considering that production gets to a halt when both particles get Boltzmann suppressed at $x = m_B/T \simeq \mathcal{O}(1)$

$\rightarrow$ the lowest possible T is $T \simeq m_B$

$\Rightarrow$ We expect $\quad Y_X \sim \dfrac{\Gamma_{B\to X}\, M_P}{m_B^2}$.

1.2. Boltzmann equations.

In this case we can go back to.

$$\frac{df_X}{dt} = \frac{1}{E_X} \, C[f_X].$$

Assuming that there is no initial density of DM $n_X(t_i) = 0$, we can neglect the reverse process $X \to B$ and write.

$$\frac{1}{E_X} C[f_X] = \frac{1}{2E_X} \int \prod_\alpha \left(\frac{d p_\alpha}{(2\pi)^3 2E_\alpha} \right) (2\pi)^4 \delta^4 (P_{fin} + p_X - P_{in})$$

ampl squared averaged over in states summed over final states. $\quad |\mathcal{M}|^2_{in \to fin + X} \; f_{in} \left(1 \pm \frac{f_X}{g_X}\right)\left(1 \pm \frac{f_{fin}}{g_{fin}}\right)$

Let us emphasize that in the case of FI it has been shown that spin-statistics can affect the results. Here however, as in the case of WIMPS, we will neglect the $(1 \pm f_i/g_i)$ factors., see e.g. 1801.03509 & micrOMEGAS.

In the latter case, we can again integrate our both side of the equation over the DM 3 momenta and we obtain :

decay width $B \to X$

$$\frac{dn_X}{dt} + 3H n_X \simeq n_B^{eq} \, \Gamma_{B \to X} \frac{K_1[m_B/T]}{K_2[m_B/T]}$$

$K_n \equiv$ Modified Bessel function of the nth kind.

$$\begin{cases} K_1[x] = x \int_1^\infty (t^2-1)^{1/2} \exp(-tx) \\ K_2[x] = x \int_1^\infty (t^2-1)^{1/2} \, t \exp(-tx) \end{cases}$$

when considering N.R. bath particle, using $f_B^{eq} = g_B \exp(-E_B/T)$

We also see that defining the time variable

$$\boxed{x = \frac{m_B}{T}}$$

is more convenient in the case of FI as the DM production will become exponentially suppressed as the bath particle becomes non relativistic. Here again going from time to temperature, using entropy conservation, we use

$$\frac{d\ln T}{dt} = -\bar{H} \quad \underline{\text{and}}$$

$$\frac{dY_X}{dx} = \frac{\Gamma_{B \to X}}{x\bar{H}} Y_B^{eq} \frac{K_1[x]}{K_2[x]}$$

Assuming constant g_* and h_{eff} over DM production, one gets

$$Y_X^\infty = \frac{405\sqrt{5}}{16\pi^{9/2}} \frac{g_B}{h_{eff}\, g_*^{1/2}} \underbrace{\frac{\Gamma_{B\to X} M_p}{m_B^2}}.$$

we recover the rule of thumb dependence.

$$\rightarrow \Omega_X h^2 = \frac{Y_X^\infty s_0 m_X}{\rho_c/h^2}$$

$$\simeq 0.12 \left(\frac{m_X}{10\,\mathrm{keV}}\right)\left(\frac{1\,\mathrm{TeV}}{m_B}\right)^2\left(\frac{g_B \Gamma_{B\to X}}{5\,10^{-15}\,\mathrm{GeV}}\right)$$

here we use $g_* = h_{eff} \simeq 100$

Introducing the convenient variable

$$R_\Gamma^{FI} = \frac{\Gamma_{B\to X} M_0}{m_B^2} \quad ; \quad M_0 = \sqrt{\frac{45}{4\pi^3 g_*(FI)}} \times M_p$$

that measure the importance of the B decay rate w.r.t. the Hubble rate in a RD era.

$$\rightarrow \Omega_X h^2\Big|_{FI\,dec} = m_X \frac{135}{8\pi^3} \frac{g_B}{h_{eff}(x_{FI})} R_\Gamma^{FI} \frac{s_0}{\rho_c/h^2}$$

$$\simeq 0.12 \left(\frac{m_X}{10\,\mathrm{keV}}\right)\left(\frac{R_\Gamma^{FI}}{3\,10^{-3}}\right)$$

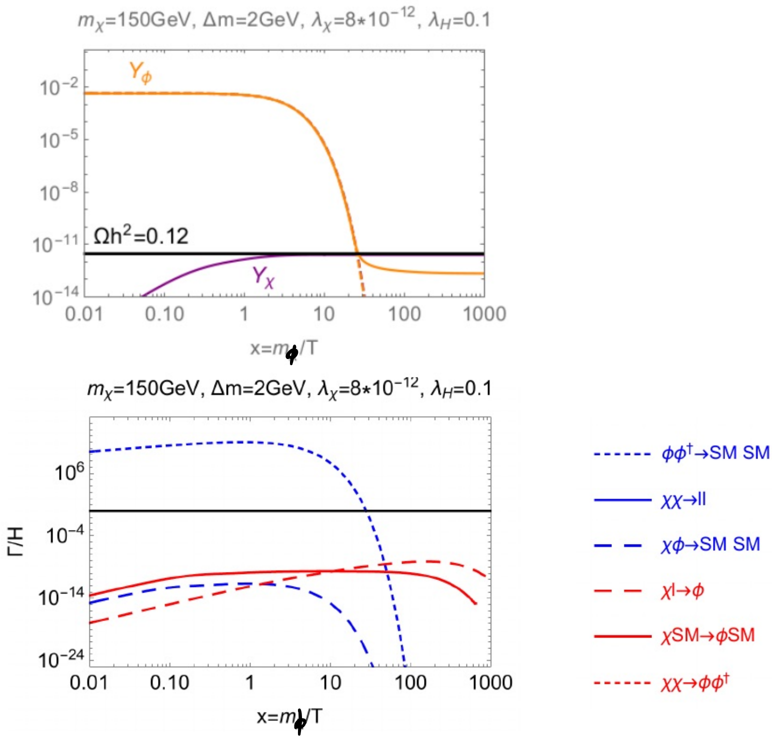
- Considering the case of e.g $m_B \gg m_X, m_{SM}$ in $B \to X\,SM$, one has
  $\Gamma_{B\to X} = \frac{\lambda_X^2}{8\pi} m_B$. i.e. $\lambda_X \sim 8\,10^{-9} \ll g$ SU(2)$_L$ coupl.
  $m_B \sim 1$TeV
  In order to account for all DM, ie, the DM is <u>much more feably coupled</u> than in the case of WIMP.

- DM is produced around the time at which B abundance gets exponentially suppressed i.e. $T \simeq m_B$

  $$\rightarrow \quad x_{FI} = \frac{m_B}{T_{FI}} \sim 1$$

1.3. Illustration.

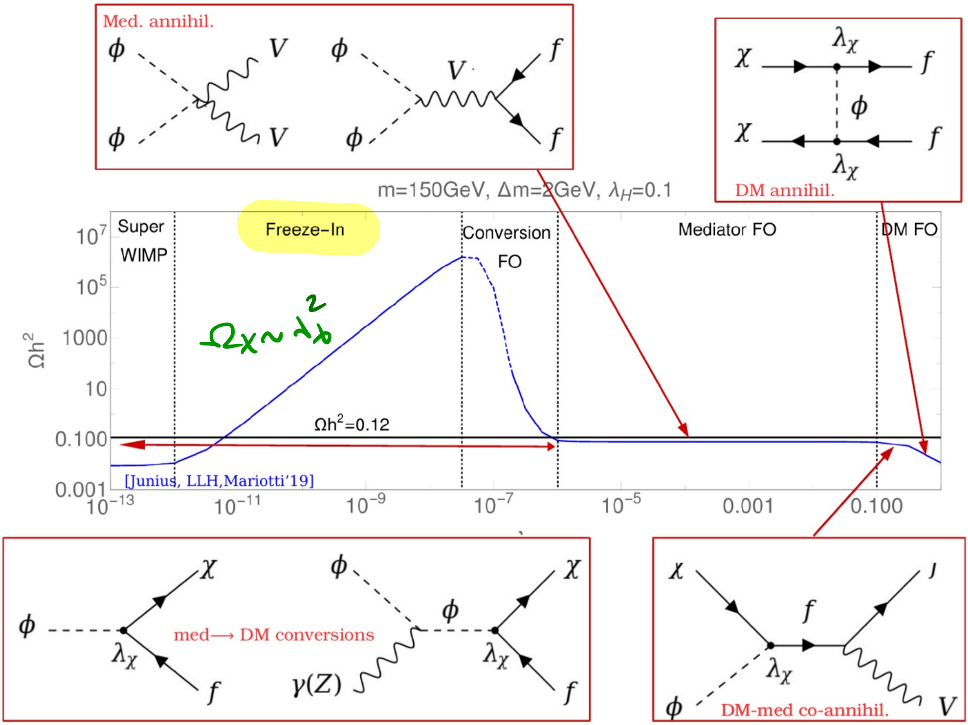
Below some illustrations in the case of leptophilic DM.



All processes except for mediator annihilation are slow compared to Hubble rate.

For the set of masses below $m_\chi, m_\phi \sim 100$ GeV we produce DM through FI for λ_χ between 10^{-12} and 10^{-8}

$$\Omega_\chi h^2 \sim \Gamma_{B\to\chi} \sim \lambda_\chi^2$$



NB: • FI from annihilation processes, see e.g. 2111.14871 for an efficient treatment
• FI in different universes histories see

2. The Super WIMP (SW) mechanism

considering even lower coupling for DM-mediator-SM, we end up with very low decay rate $\Leftrightarrow$ long life time of the bath particle. very few DM would be produced at early time through FI

- In the latter case, the bath particle could have gone through Freeze-out and

$$Y_B = Y_B^{FO} \quad \text{for} \quad x_f^B < x < x_{SW}.$$

in particular, this means that B has developed a non-neglibible chem. potential

$$\chi f_B^{eq}$$

$$f_{in} = f_B = \frac{f_B^{eq}}{Y_B^{eq}} Y_B$$

when $H \sim \Gamma_B$

At late time, the B-particle decays fully to DM. We can thus expect

$$Y_B^{FO} = Y_X^{\infty} \quad \Longrightarrow \quad \boxed{\Omega_X^{SW} h^2 \sim \frac{m_X}{m_B} \Omega_B h^2}$$

- We can recover $\Omega_X h^2$ from the same Boltzman eqs.

$x > x_{FO}^B$

i.e. $Y_B \neq Y_B^{eq}$

$$\begin{cases} \dfrac{dY_B}{dx} = - \underbrace{\dfrac{\Gamma_{B\to X}}{\bar{H}\, x}}_{x \frac{\Gamma_{B\to X} M_0}{m_B^2}} \underbrace{\dfrac{K_1(x)}{K_2(x)}}_{x > x_f \text{ ie } x \gg 1} Y_B \\ \\ \dfrac{dY_X}{dx} = \dfrac{\Gamma_{B\to X}}{x \bar{H}} \dfrac{K_1[x]}{K_2[x]} Y_B . \end{cases}$$

$K_1[x]/K_2[x] \to 1$

- For the B particle, using

$Y_B = Y_B^{FO} = \text{cst}$. at $x = x_{FO}^B$, and $g_* = \text{cst}$

$$\Rightarrow \boxed{Y_B(x) = Y_B^{FO} \, e^{-R_\Gamma (x^2 - x_{FO}^2)/2}} \quad x \overset{B}{>} x_{FO}$$

(**)

with $R_\Gamma^{SW} = \dfrac{\Gamma_{B\to X} M_0}{m_B^2}$; $M_0 = \sqrt{\dfrac{45}{4\pi^3 g_*(x_{SW})}} \times M_P$

(**) $\Rightarrow$ the decay time, or x_{SW}, for $x_{SW} \gg x_{FO}$ is:

$$\boxed{x_{SW}^2 = \left(\frac{R_\Gamma^{SW}}{2}\right)^{-1}} \iff \frac{m_B^2}{T_{SW}^2} = \frac{2\, m_B^2}{\Gamma_B M_0}.$$

$$\iff \frac{T_{SW}^2}{M_0} \sim H(T_{SW}) \sim \Gamma_B/2 \quad \checkmark$$

- For the χ particle you have:

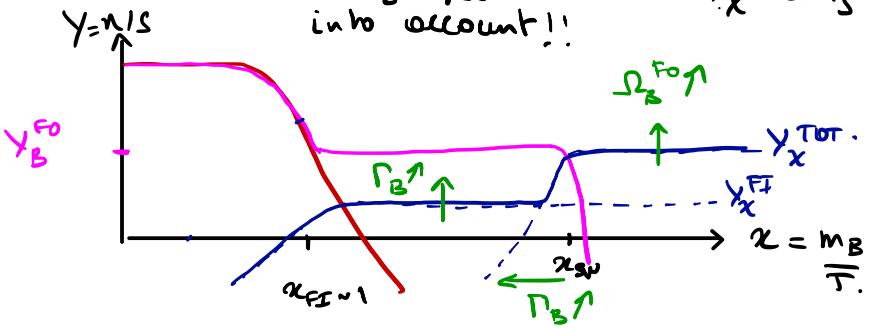
$$Y_\chi(x_{SW}) = Y_B^{FO} \cdot R_\Gamma \int_0^\infty dx\, x\, e^{-R_\Gamma (x^2 - x_{FO}^2)/2}$$

$$= Y_B^{FO} \exp\left(x_{FO}^2 / x_{SW}^2\right) \to Y_B^{FO}$$

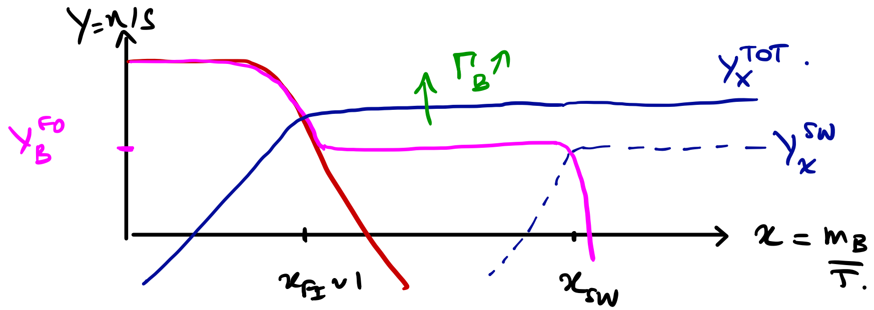
for $x_{SW} \gg x_{FO}$.

as expected!

- You always have to take Y_χ^{FI} & Y_S^{SW} into account!!



or



- As a final comment

$$Y_\chi^\infty = Y_B^{FO} \Rightarrow \Omega_\chi^{SW} h^2 \propto Y_B^{FO} \propto \frac{1}{\sigma_{22}} \propto \frac{1}{g^2}$$

which is independent of λ_χ

3. Conversion driven F.O.

In V, we have seen that we can go from the vanilla WIMP case involving couplings $\sim g_{SU(2)_L}$ to the case where the DM would get its relic abundance fixed by the Z_2 odd partner if the relative mass splitting is small (coannihilation driven)

Now for 3 body interaction $\mathcal{L} \supset \lambda_\chi B \chi A_{SM}$

the dependence $\left(\Delta = (m_\phi - m_\chi)/m_\chi \right)$

$$\Omega_\chi h^2 \propto \frac{1}{\langle \sigma v_{eff} \rangle} \propto \begin{cases} \lambda_\chi^{-4} & \text{DM FO} \\ \lambda_\chi^{-2} g^{-2} \, 1/\exp(-\Delta x) & \text{CO-ANN FO.} \\ g^{-4} \, 1/\exp(-2\Delta x) & \text{MEDIATOR FO.} \end{cases}$$

assumes that $B \leftrightarrow \chi$ conversion processes happen fast enough to ensure chemical equilibrium and as a result

$$\frac{n_B}{n_B^{eq}} = \frac{n_\chi}{n_\chi^{eq}}$$

It has however been noticed though, see 1705.09292, 1705.084600, that when conversion processes get suppressed enough so that $\frac{\gamma_{i \leftrightarrow j}}{H n_i} < 1$

where $\gamma_{i \leftrightarrow j}$ is the reaction rate associated to $\chi_i \leftrightarrow \chi_j$ conversions, the departure from chemical equilibrium re-introduces a λ_χ dependence in the $\Omega_\chi h^2$ for $\lambda_\chi \ll 1$ and $\Delta_{ij} = \frac{m_i - m_j}{m_i} < 1$.

In order to compute correctly the DM abundance, one should definitively account for $\gamma_{i \leftrightarrow j}$ effects on the $\{Y_i\}_{i=1..N}$

For the framework we are concerned with considering $\chi_1 = \chi$ and $\chi_2 = B$, we should solve the coupled Boltzmann system:

$$\frac{dY_1}{dx} = \frac{-1}{Hx}\left[\gamma_{11}\left(\frac{Y_1^2}{Y_{1eq}^2} - 1\right) + \gamma_{12}\left(\frac{Y_1 Y_2}{Y_{1eq} Y_{2eq}} - 1\right) \boxed{- \gamma_{2\to1}\left(\frac{Y_2}{Y_{2eq}} - \frac{Y_1}{Y_1^{eq}}\right)} + \gamma_{11\to22}\left(\frac{Y_1^2}{Y_{1eq}^2} - \frac{Y_2^2}{Y_{2,eq}^2}\right)\right]$$

$$\frac{dY_2}{dx} = \frac{-1}{Hx}\left[\gamma_{22}\left(\frac{Y_1^2}{Y_{1eq}^2} - 1\right) + \gamma_{12}\left(\frac{Y_1 Y_2}{Y_{1eq} Y_{2eq}} - 1\right) \boxed{+ \gamma_{2\to1}\left(\frac{Y_2}{Y_{2eq}} - \frac{Y_1}{Y_1^{eq}}\right)} - \gamma_{11\to22}\left(\frac{Y_1^2}{Y_{1eq}^2} - \frac{Y_2^2}{Y_{2,eq}^2}\right)\right]$$

Caution : in order to write the above equations, we are assuming that χ is in kinetic equ. with B. This can not be ensured for arbitrarily small couplings !

It has been shown that for the parameter space testable by experiments with a B ≡ colored particle kinetic equilibrium is a good approach, see 1705.09292. In some other cases, where the coupling involved in coscattering processes is smaller, it is necessary to go back to the unintegrated Boltzmann eqs. see e.g. 1705.08450.

Here, we describe the case where departure from kinetic equilibrium can be neglected and the use of the integrated Boltzmann equ. can be trusted*

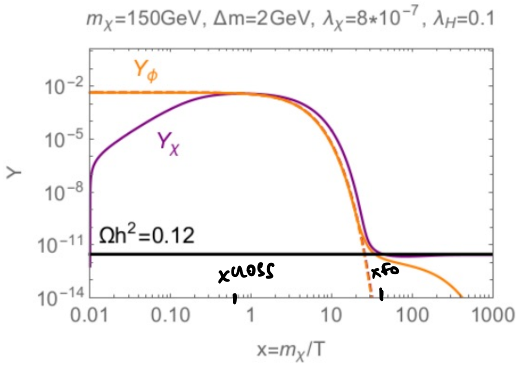
* in the $(\lambda_\chi, \Omega_\chi h^2)$ plot for leptophilic DM, we have used a dashed line to emphasize the region where we do not expect kinetic eq.

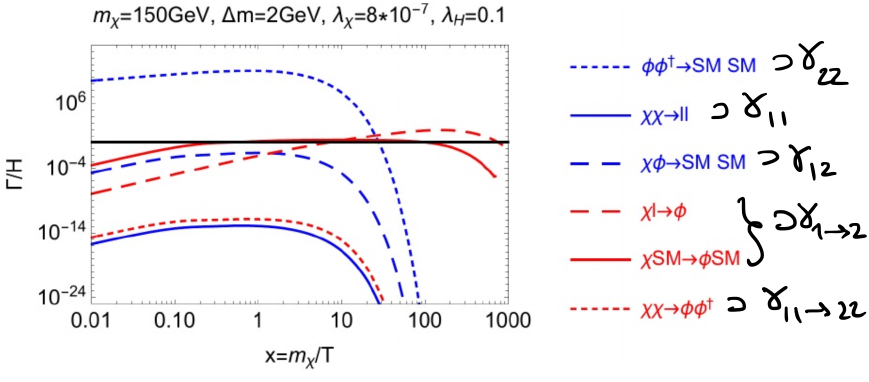
In the framework of conversion driven F.O. we can not provide analytic results but numerical results.

We have thus to discuss a specific model, the leptophilic DM case:

$$\mathcal{L} \supset \lambda_\chi \bar{\chi} e_R \phi$$

Below, we show the DM and mediator density evolution for $m_{DM} = 150\,\text{GeV}$, $\Delta m_{\chi\phi} = 2\,\text{GeV}$ and $\lambda_\chi = 8 \cdot 10^{-7}$





Taking $Y_\chi(x=0)=0$ as initial condition $Y_\chi(x)$ slowly builds up as in the case of FI. This time though $Y_\chi(x)$ is going to cross $Y_\chi^{eq}(x)$ before it freezes-out.

In contrast, ϕ is maintained in chemical equil with the plasma thanks to its gauge interactions.
On the other hand $\chi \to \phi$ conversions are barely efficient as can be seen in the bottom plot with red colors:

$$\Gamma_{\chi\to\phi} = \frac{\gamma_{\chi\to\phi}}{n_\chi^{eq}} \lesssim H$$

Such barely efficient conversion processes maintain: $Y_\chi(x) \gtrsim Y_\chi^{eq}(x)$ $\quad x_{cross} < x < x_{FO}$

As a result, once the mediator gets out of chemical equilibrium at $x_{FO}^B \simeq 25$ and eventually decays to χ, the DM density freezes out at slightly later time.

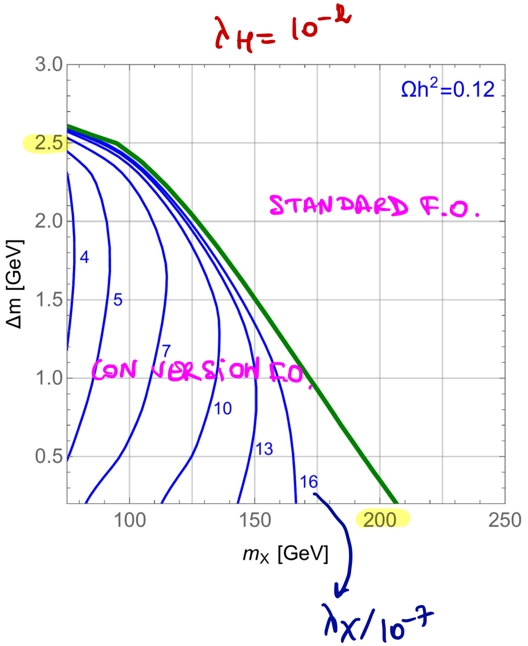
Now, the smaller is λ_χ, the more important will be the departure of $Y_\chi(x)$ compared to Y_χ^{eq} and the larger $Y_\chi(x_f)$ becomes.

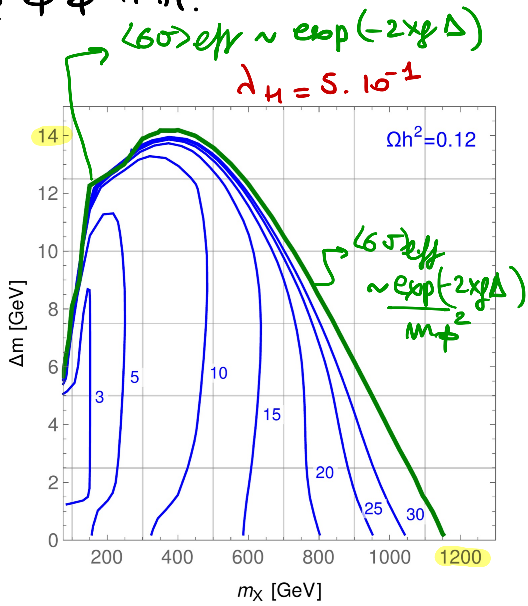
This behavior of $\left\{ \begin{matrix} Y_\chi \\ \Omega_\chi h^2 \end{matrix} \right. \nearrow$ for $\lambda_\chi \searrow$ is well visible in our $(\lambda_\chi, \Omega_\chi h^2)$ plot.

One last comment. If we consider additional $BB \to SMSM$ processes, we can actually <u>decrease</u> the expected $\Omega_X h^2$ that would be expected from mediator annihilation. This also implies that a larger $(m_B, \Delta m)$ parameter space is able to give rise to conversion driven FO.

The reason for this is because if $\sigma_{22} \uparrow \Rightarrow \Omega_X h^2|_{\text{med ann}} \downarrow$
$\Rightarrow$ more parameter space to compensate with λ_X due to inefficient conversions.

In the case of dephrophic DM, an additional annihilation channel is provided by considering $\mathcal{L} \supset \lambda_H \phi^\dagger \phi H^\dagger H$.





Green limit of the parameter space corresponds to $\Omega_\chi h^2 = 0.12$ due to mediator annihilation F.O. i.e. for
$\langle \sigma v_{eff} \rangle \propto \langle \sigma v_{22} \rangle \exp(-2 x_f \Delta)$. with $\Delta = \frac{\Delta m_{\chi\phi}}{m_\chi}$

The form of the latter can be easily understood as a competition between the mediator annihilation cross section $\langle \sigma v \rangle_{22}$ and the Boltzmann suppression factor $\exp(-2 x_f \Delta)$, see the plot.

NB In the case of colored mediator a careful treatment of non perturbative effects is needed in order to account correctly for the bath particle abundance in the all process, see 2112.01499.